54.5

Consider the covering map \( p \times p : \mathbb{R} \times \mathbb{R} \to S^1 \times S^1 \) of Example 53.4. Consider the path

\[
f(t) = (\cos 2\pi t, \sin 2\pi t) \times (\cos 4\pi t, \sin 4\pi t)
\]

in \( S^1 \times S^1 \). Sketch what \( f \) looks like when \( S^1 \times S^1 \) is identified with the doughnut surface \( D \). Find a lifting \( \tilde{f} \) of \( f \) to \( \mathbb{R} \times \mathbb{R} \) and sketch it.

**Answer:** See attached sheet for sketches. Let \( \tilde{f}(t) = t \times 2t \). Then

\[
((p \times p)(\tilde{f}))(t) = (p \times p)(t \times 2t) = (\cos 2\pi t, \sin 2\pi t) \times (\cos 4\pi t, \sin 4\pi t) = f(t),
\]

so \( \tilde{f} \) is a lifting of \( f \). ♣

54.7

Generalize the proof of Theorem 54.5 to show that the fundamental group of the torus is isomorphic to the group \( \mathbb{Z} \times \mathbb{Z} \).

**Proof.** Let \( p \times p : \mathbb{R} \times \mathbb{R} \to S^1 \times S^1 \) be as in 54.5, let \( e_0 = (0, 0) \) and let \( b_0 = p(e_0) \). Then \( p^{-1}(b_0) \) is the set \( \mathbb{Z} \times \mathbb{Z} \). Since \( \mathbb{R}^2 \) is simply connected, the lifting correspondance

\[
\phi : \pi_1(S^1 \times S^1; b_0) \to \mathbb{Z} \times \mathbb{Z}
\]

is bijective. We show that \( \phi \) is a homomorphism, which gives us the desired result.

Given \([f], [g] \in \pi_1(S^1 \times S^1; b_0)\), let \( \tilde{f} \) and \( \tilde{g} \) be their respective liftings to paths on \( \mathbb{R}^2 \) beginning at \( e_0 \). Let \((n_1, n_2) = \tilde{f}(1) \) and \((m_1, m_2) = \tilde{g}(1)\); then \( \phi([f]) = (n_1, n_2) \) and \( \phi([g]) = (m_1, m_2) \), by definition. Let \( \tilde{g} \) be the path

\[
\tilde{g}(s) = (n_1, n_2) + \tilde{g}(s)
\]

on \( \mathbb{R}^2 \). Since \( p((n_1, n_2) + (x_1, x_2)) = p((x_1, x_2)) \) for all \((x_1, x_2) \in \mathbb{R}^2\), the path \( \tilde{g} \) is a lifting of \( g \); it begins at \((n_1, n_2) \). Then \( f \ast \tilde{g} \) is defined, and is the lifting of \( f \ast g \) that begins at \((n_1, n_2) \). The end point of this path is \( \tilde{g}(1) = (n_1, n_2) + (m_1, m_2) \). Then, by definition

\[
\phi([f] \ast [g]) = (n_1 + m_1, n_2, m_2) = \phi([f]) + \phi([g]).
\]

\(\square\)
Show that if $A$ is a nonsingular $3 \times 3$ matrix having nonnegative entries, then $A$ has a positive real eigenvalue.

**Proof.** Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation whose matrix, relative to the standard basis for $\mathbb{R}^3$, is $A$. Let $B$ be the intersection of the 2-sphere $S^2$ with the first octant of $\mathbb{R}^3$. $B$ is homeomorphic to the ball $B^2$, so the Brouwer fixed-point theorem holds for continuous maps of $B$ into itself.

Now, if $x = (x_1, x_2, x_3) \in B$, then all components of $x$ are nonnegative and at least one is positive. Since all entries of $A$ are non-negative, the vector $T(x)$ is a vector with all nonnegative components. Furthermore, since $A$ is non-singular, its nullspace is trivial, so $x \notin \text{Nul} A$, meaning $T(x) \neq 0$. Therefore, $||T(x)|| \neq 0$, so the map $x \mapsto T(x)/||T(x)||$ is a continuous map of $B$ into itself. By the fixed-point theorem, then, this map has a fixed point $x_0$. Then

$$T(x_0) = ||T(x_0)||x_0,$$

so $T$ (and, hence, $A$) has a positive real eigenvalue $||T(x_0)||$. \qed

**Definition 0.1.** Let $G$ be a group and $X$ a set. A (right) group action on $X$ is a map $X \times G \to X$ given by $(x, g) \mapsto x \cdot g$, such that

i) $x \cdot e = x$ for all $x \in X$.

ii) $x(g_1g_2) = (x \cdot g_1) \cdot g_2$ for any $x \in X, g_1, g_2 \in G$.

Now, let $p : E \to B$ be a covering map and fix $b_0 \in B$. Now let $p^{-1}(b_0) \times \pi_1(B; b_0) \to p^{-1}(b_0)$ be given by $x \cdot [f] = \tilde{f}(1)$, where $\tilde{f} : I \to E$ is the unique lift of $f$ to a path based at $x \in p^{-1}(b_0)$.

(a) Show that this is a well-defined right group action of $\pi_1(B; b_0)$ on the fiber $p^{-1}(b_0)$. It is sometimes referred to as the monodromy action.

**Proof.** Let $[1] \in \pi_1(B; b_0)$ be the identity element. Then, if $x \in p^{-1}(b_0)$,

$$x \cdot [1] = \tilde{1}(1) = x,$$

since $\tilde{1}$ is just the trivial loop based at $x$. Furthermore, if $[f], [g] \in \pi_1(B; b_0)$ and $x \in p^{-1}(b_0)$,

$$x \cdot ([f] \ast [g]) = x \cdot [f \ast g] = \tilde{f} \ast \tilde{g}(1),$$

where $\tilde{f} \ast \tilde{g}$ is the unique lift of $f \ast g$ to a path based at $x$. Now, on the other hand,

$$(x \cdot [f]) \cdot [g] = \tilde{f}(1) \cdot [g] = \tilde{g}(1),$$

where $\tilde{g}$ is the unique lift of $g$ to a path based at $\tilde{f}(1)$. Now, note that

$$\tilde{g}(1) = \tilde{f} \ast \tilde{g}(1),$$

so $x \cdot ([f] \ast [g]) = (x \cdot [f]) \cdot [g]$, so this is a well-defined group action. \qed
(b) Show that if $E$ is path connected this action is transitive. That is, for any $x, y \in p^{-1}(b_0)$ there exists $[\alpha] \in \pi_1(B; b_0)$ such that $x \cdot [\alpha] = y.$

**Proof.** Let $x, y \in p^{-1}(b_0).$ Then, since $E$ is path-connected, there exists a path $\overline{x}$ from $x$ to $y.$ Now, define $\alpha = p \circ \overline{x}.$ Then

$$\alpha(0) = (p \circ \overline{x})(0) = p(\overline{x}(0)) = p(x) = b_0,$$

$$\alpha(1) = (p \circ \overline{x})(1) = p(\overline{x}(1)) = p(y) = b_0,$$

so $\alpha$ is a loop based at $b_0.$ Furthermore, since the lift of $\alpha$ to a path based at $x$ is unique, and $\overline{x}$ is certainly a lift of $\alpha$ based at $x,$ we see that

$$x \cdot [\alpha] = \overline{x}(1) = y.$$  

\qed

(c) Given a (right) action of some group $G$ on a set $X,$ the isotropy subgroup of $G$ corresponding to $x_0 \in X$ is the subgroup $G_{x_0} = \{ g \in G : x_0 \cdot g = x_0 \}.$ Now, given $x_0 \in p^{-1}(b_0),$ what is the corresponding isotropy subgroup of the monodromy action?

**Answer:** Suppose $[f] \in G_{x_0}.$ Then $f = p \circ \tilde{f}$ where $\tilde{f}$ is the unique lift of $f$ to a path in $E$ based at $x_0$ such that $\tilde{f}(1) = x_0.$ This implies that $\tilde{f} \in \Omega(E, x_0),$ so $[\tilde{f}] \in \pi_1(E; x_0).$ Hence,

$$G_{x_0} \subseteq \{ [p \circ \tilde{f}] \in \pi_1(B; b_0) : [\tilde{f}] \in \pi_1(E; x_0) \}.$$  

On the other hand, if $[\tilde{f}] \in \pi_1(E; x_0),$ then

$$x_0 \cdot [p \circ \tilde{f}] = (p \circ \tilde{f})(1) = \tilde{f}(1) = x_0,$$

since $p \circ \tilde{f} = p \circ \tilde{f}$ and this lifting is unique. Hence,

$$\{ [p \circ \tilde{f}] \in \pi_1(B; b_0) : [\tilde{f}] \in \pi_1(E; x_0) \} \subseteq G_{x_0}.$$  

Therefore, we can conclude that $G_{x_0} = \{ [p \circ \tilde{f}] \in \pi_1(B; b_0) : [\tilde{f}] \in \pi_1(E; x_0) \}.$  

\medskip

**B**

For each $n \in \mathbb{N},$ let $p_n : S^1 \to S^1$ be given by $z \mapsto z^n,$ where $S^1 = \{ z \in \mathbb{C} : ||z|| = 1 \}.$

(a) Show that $p_n$ is a covering map.

**Proof.** Fix $n \in \mathbb{N}.$ Then $p_n$ is certainly continuous and surjective, so we need only check that each point in $S^1$ has a neighborhood that is evenly covered by $p_n$ to show that $p_n$ is a covering map. Consider the open subset $U$ consisting of all points on the circle having at least one positive coordinate. We can write

$$U = \{ e^{i\theta} | -\pi/2 < \theta < \pi \}.$$  

Then

$$p_n^{-1}(U) = \{ e^{i\frac{2\pi k + \theta}{n}} | -\pi/2 < \theta < \pi, k = 0, 1, \ldots (n - 1) \}.$$  

If we let $V_k = \{e^{i \frac{2\pi k + \theta}{n}} \mid -\pi/2 < \theta < \pi\}$, then we see that
\[
p^{-1}(U) = \bigcup_{k=0}^{n-1} V_k.
\]
Now, we need to show that $p|_{V_k} \to U$ is a homeomorphism for each $k$. Fix $j \in \{0, 1, \ldots, (n-1)\}$. Continuity is clear. Now, if $x = e^{i \frac{2\pi j + \theta_1}{n}} \in V_j$ such that $x^n = p|_{V_j}(x) = p|_{V_j}(y) = y^n$, then
\[
x^n = \left(e^{i \frac{2\pi j + \theta_1}{n}}\right)^n = e^{2\pi j \theta_1} = e^{i \theta_1}
\]
and
\[
y^n = \left(e^{i \frac{2\pi j + \theta_2}{n}}\right)^n = e^{2\pi j \theta_2} = e^{i \theta_2}
\]
so $\theta_1 = \theta_2$, meaning $x = y$. Hence, $p|_{V_j}$ is injective.

If $e^{i \theta} \in U$, then $x = e^{i \frac{2\pi j + \theta}{n}} \in V_j$ and $p|_{V_j}(x) = e^{i \theta}$, so $p|_{V_j}$ is surjective.

Now, it suffices to show that $p|_{V_j} : \overline{V_j} \to \overline{U}$ is a homeomorphism. $p|_{V_j}$ is bijective for the same reasons $p|_{V_j}$ is. Also, if $C \subseteq \overline{V_j}$ is closed, then, since $\overline{V_j}$ is compact, $C$ is compact, so its image under $p|_{V_j}$ is compact and thus, since $\overline{U}$ is Hausdorff, closed. Hence $p|_{V_j} : \overline{V_j} \to \overline{U}$ is a closed, bijective map and, therefore, a homeomorphism. Therefore, $p|_{V_j} : V_j \to U$ is a homeomorphism and so we see that $U$ is evenly covered by $p$.

A similar argument demonstrates that the subset $U' \subseteq S^1$ consisting of all points on the circle having at least one negative coordinate is evenly covered by $p$. Every element in $S^1$ is contained in either $U$ or $U'$ (or both), so every element as a neighborhood evenly covered by $p$, to $p$ is a covering map. □

(b) What are all the non-trivial subgroups of $\mathbb{Z}$?

**Answer:** The non-trivial subgroups of $\mathbb{Z}$ are those groups of the form
\[
n\mathbb{Z} = \{na : a \in \mathbb{Z}\}
\]
for some $n \in \mathbb{Z}$.

To see this, let $G$ be a subgroup of $\mathbb{Z}$. Since $\mathbb{Z}$ is well-ordered, there exists some positive, smallest $g \in G$. Let $x \in G$, $x \neq 0$. Then, by the Euclidean Algorithm, there exist $q, r \in \mathbb{Z}$ such that
\[
x = qg + r
\]
where $0 \leq r < g$. Now,
\[
qg = \underbrace{g + g + \ldots + g}_{q \text{ summands}}
\]
so $qg \in G$. Hence, since $x \in G$, $r = x - qg \in G$. Since $g$ is the minimal positive element of $G$, this means it must be the case that $r = 0$. Therefore, every non-zero element of $G$ is a multiple of $g$, so $G = n\mathbb{Z}$. ♣
(c) What subgroup of $\mathbb{Z}$ is $p_n^*(\pi_n(S^1;1))$ isomorphic to?

**Answer:** Since $\mathbb{Z}$ and all of its subgroups are cyclic, this isomorphism will be completely determined by its action on the generator of $\pi_1(S^1;1)$, namely $[\sigma]$ where $\sigma : [0,1] \to S^1$ is given by $\sigma(t) = e^{i2\pi t}$. Now, 

$$p_n^*([\sigma]) = [p_n \circ \sigma]$$

where

$$(p_n \circ \sigma)(t) = p_n(\sigma(t)) = p_n(e^{i2\pi t}) = (e^{i2\pi t})^n = e^{i2\pi nt}.$$ 

If $\phi : \pi_1(S^1;1) \to \mathbb{Z}$ is the natural isomorphism, then $\phi(e^{i2\pi nt}) = n$. In other words, $p_n^*([\sigma])$ corresponds to the element $n \in \mathbb{Z}$. Hence, $p_n^*(\pi_1(S^1;1))$ is isomorphic to $n\mathbb{Z}$.

♣

DRL 3E3A, University of Pennsylvania

*E-mail address: shonkwil@math.upenn.edu*