TOPOLOGY HW 7

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54.5

Consider the covering map $p \times p : \mathbb{R} \times \mathbb{R} \to S^1 \times S^1$ of Example 53.4. Consider the path

$$f(t) = (\cos 2\pi t, \sin 2\pi t) \times (\cos 4\pi t, \sin 4\pi t)$$

in $S^1 \times S^1$. Sketch what f looks like when $S^1 \times S^1$ is identified with the doughnut surface D. Find a lifting $\tilde{f}$ of f to $\mathbb{R} \times \mathbb{R}$ and sketch it.

Answer: See attached sheet for sketches. Let $\tilde{f}(t) = t \times 2t$. Then

$$((p\times p)\circ\tilde{f})(t) = (p\times p)(\tilde{f}(t)) = (p\times p)(t\times 2t) = (\cos 2\pi t, \sin 2\pi t)\times(\cos 4\pi t, \sin 4\pi t) = f(t),$$

so $\tilde{f}$ is a lifting of f.

♣

54.7

Generalize the proof of Theorem 54.5 to show that the fundamental group of the torus is isomorphic to the group $\mathbb{Z} \times \mathbb{Z}$.

Proof. Let $p \times p : \mathbb{R} \times \mathbb{R} \to S^1 \times S^1$ be as in 54.5, let $e_0 = (0,0)$ and let $b_0 = p(e_0)$. Then $p^{-1}(b_0)$ is the set $\mathbb{Z} \times \mathbb{Z}$. Since $\mathbb{R}^2$ is simply connnected, the lifting correspondance

$$\phi : \pi_1(S^1 \times S^1; b_0) \to \mathbb{Z} \times \mathbb{Z}$$

is bijective. We show that ϕ is a homomorphism, which gives us the desired result.

Given $[f], [g] \in \pi_1(S^1 \times S^1; b_0)$, let $\tilde{f}$ and $\tilde{g}$ be their respective liftings to paths on $\mathbb{R}^2$ beginning at e_0. Let $(n_1, n_2) = \tilde{f}(1)$ and $(m_1, m_2) = \tilde{g}(1)$; then $\phi([f]) = (n_1, n_2)$ and $\phi([g]) = (m_1, m_2)$, by definition. Let $\tilde{\tilde{g}}$ be the path

$$\tilde{\tilde{g}}(s) = (n_1, n_2) + \tilde{g}(s)$$

on $\mathbb{R}^2$. Since $p((n_1, n_2) + (x_1, x_2)) = p((x_1, x_2))$ for all $(x_1, x_2) \in \mathbb{R}^2$, the path $\tilde{\tilde{g}}$ is a lifting of g; it begins at (n_1, n_2). Then $f * \tilde{\tilde{g}}$ is defined, and is the lifting of $f * g$ that begins at 0. The end point of this path is $\tilde{\tilde{g}}(1) = (n_1, n_2) + (m_1, m_2)$. Then, by definition

$$\phi([f] * [g]) = (n_1 + m_1, n_2, m_2) = \phi([f]) + \phi([g]).$$

□

55.3

Show that if A is a nonsingular 3×3 matrix having nonnegative entries, then A has a positive real eigenvalue.

Proof. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation whose matrix, relative to the standard basis for $\mathbb{R}^3$, is A. Let B be the intersection of the 2-sphere S^2 with the first octant of $\mathbb{R}^3$. B is homeomorphic to the ball B^2, so the Brouwer fixed-point theorem holds for continuous maps of B into itself.

Now, if $x = (x_1, x_2, x_3) \in B$, then all components of x are nonnegative and at least one is positive. Since all entries of A are non-negative, the vector $T(x)$ is a vector with all nonnegative components. Furthermore, since A is non-singular, its nullspace is trivial, so $x \notin \mathrm{Nul}A$, meaning $T(x) \neq 0$. Therefore, $||T(x)|| \neq 0$, so the map $x \mapsto T(x)/||T(x)||$ is a continuous map of B into itself. By the fixed-point theorem, then, this map has a fixed point x_0. Then

$$T(x_0) = ||T(x_0)||x_0,$$

so T (and, hence, A) has a positive real eigenvalue $||T(x_0)||$. □

A

Definition 0.1. Let G be a group and X a set. A **(right) group action** on X is a map $X \times G \to X$ given by $(x, g) \mapsto x \cdot g$, such that

i) $x \cdot e = x$ for all $x \in X$.

ii) $x(g_1g_2) = (x \cdot g_1) \cdot g_2$ for any $x \in X, g_1, g_2 \in G$.

Now, let $p : E \to B$ be a covering map and fix $b_0 \in B$. Now let $p^{-1}(b_0) \times \pi_1(B; b_0) \to p^{-1}(b_0)$ be given by $x \cdot [f] = \tilde{f}(1)$, where $\tilde{f} : I \to E$ is the unique lift of f to a path based at $x \in p^{-1}(b_0)$.

(a) Show that this is a well-defined right group action of $\pi_1(B; b_0)$ on the fiber $p^{-1}(b_0)$. It is sometimes referred to as the *monodromy action.*

Proof. Let $[1] \in \pi_1(B; b_0)$ be the identity element. Then, if $x \in p^{-1}(b_0)$,

$$x \cdot [1] = \tilde{1}(1) = x,$$

since $\tilde{1}$ is just the trivial loop based at x. Furthermore, if $[f], [g] \in \pi_1(B; b_0)$ and $x \in p^{-1}(b_0)$,

$$x \cdot ([f] * [g]) = x \cdot [f * g] = (\widetilde{f * g})(1),$$

where $\widetilde{f * g}$ is the unique lift of $f * g$ to a path based at x. Now, on the other hand,

$$(x \cdot [f]) \cdot [g] = \tilde{f}(1) \cdot [g] = \tilde{g}(1),$$

where $\tilde{g}$ is the unique lift of g to a path based at $\tilde{f}(1)$. Now, note that

$$\tilde{g}(1) = \widetilde{f * g}(1),$$

so $x \cdot ([f] * [g]) = (x \cdot [f]) \cdot [g]$, so this is a well-defined group action. □

(b) Show that if E is path connected this action is transitive. That is, for any $x, y \in p^{-1}(b_0)$ there exists $[\alpha] \in \pi_1(B; b_0)$ such that $x \cdot [\alpha] = y$.

Proof. Let $x, y \in p^{-1}(b_0)$. Then, since E is path-connected, there exists a path $\overline{\alpha}$ from x to y. Now, define $\alpha = p \circ \overline{\alpha}$. Then

$$\alpha(0) = (p \circ \overline{\alpha})(0) = p(\overline{\alpha}(0)) = p(x) = b_0,$$
$$\alpha(1) = (p \circ \overline{\alpha})(1) = p(\overline{\alpha}(1)) = p(y) = b_0,$$

so α is a loop based at b_0. Furthermore, since the lift of α to a path based at x is unique, and $\overline{x}$ is certainly a lift of α based at x, we see that

$$x \cdot [\alpha] = \overline{\alpha}(1) = y.$$

□

(c) Given a (right) action of some group G on a set X, the isotropy subgroup of G corresponding to $x_0 \in X$ is the subgroup $G_{x_0} = \{g \in G : x_0 \cdot g = x_0\}$. Now, given $x_0 \in p^{-1}(b_0)$, what is the corresponding isotropy subgroup of the monodromy action?

Answer: Suppose $[f] \in G_{x_0}$. Then $f = p \circ \tilde{f}$ where $\tilde{f}$ is the unique lift of f to a path in E based at x_0 such that $\tilde{f}(1) = x_0$. This implies that $\tilde{f} \in \Omega(E, x_0)$, so $[\tilde{f}] \in \pi_1(E, x_0)$. Hence,

$$G_{x_0} \subseteq \{[p \circ \tilde{f}] \in \pi_1(B; b_0) : [\tilde{f}] \in \pi_1(E; x_0)\}.$$

On the other hand, if $[\tilde{f}] \in \pi_1(E; x_0)$, then

$$x_0 \cdot [p \circ \tilde{f}] = (\widetilde{p \circ \tilde{f}})(1) = \tilde{f}(1) = x_0,$$

since $p \circ \tilde{f} = p \circ \tilde{f}$ and this lifting is unique. Hence,

$$\{[p \circ \tilde{f}] \in \pi_1(B; b_0) : [\tilde{f}] \in \pi_1(E; x_0)\} \subseteq G_{x_0}.$$

Therefore, we can conclude that $G_{x_0} = \{[p \circ \tilde{f}] \in \pi_1(B; b_0) : [\tilde{f}] \in \pi_1(E; x_0)\}$.

♣

B

For each $n \in \mathbb{N}$, let $p_n : S^1 \to S^1$ be given by $z \mapsto z^n$, where $S^1 = \{z \in \mathbb{C} : ||z|| = 1\}$.

(a) Show that p_n is a covering map.

Proof. Fix $n \in \mathbb{N}$. Then p_n is certainly continuous and surjective, so we need only check that each point in S^1 has a neighborhood that is evenly covered by p_n to show that p_n is a covering map. Consider the open subset U consisting of all points on the circle having at least one positive coordinate. We can write

$$U = \{e^{i\theta} | -\pi/2 < \theta < \pi\}.$$

Then

$$p^{-1}(U) = \{e^{i\frac{2\pi k + \theta}{n}} | -\pi/2 < \theta < \pi, k = 0, 1, \dots (n-1)\}.$$

If we let $V_k = \{e^{i\frac{2\pi k+\theta}{n}} | -\pi/2 < \theta < \pi\}$, then we see that

$$p^{-1}(U) = \bigsqcup_{k=0}^{n-1} V_k.$$

Now, we need to show that $p|_{V_k} \to U$ is a homeomorphism for each k. Fix $j \in \{0, 1, \ldots (n-1)\}$. Continuity is clear. Now, if $x = e^{i\frac{2\pi j+\theta_1}{n}}, y = e^{i\frac{2\pi j+\theta_2}{n}} \in V_j$ such that $x^n = p|_{V_j}(x) = p|_{V_j}(y) = y^n$, then

$$x^n = \left(e^{i\frac{2\pi j+\theta_1}{n}}\right)^n = e^{2\pi j}e^{i\theta_1} = e^{i\theta_1}$$

and

$$y_n = \left(e^{i\frac{2\pi j+\theta_2}{n}}\right)^n = e^{2\pi j}e^{i\theta_2} = e^{i\theta_2}$$

so $\theta_1 = \theta_2$, meaning $x = y$. Hence, $p|_{V_j}$ is injective.

If $e^{i\theta} \in U$, then $x = e^{i\frac{2\pi j+\theta}{n}} \in V_j$ and $p|_{V_j}(x) = e^{i\theta}$, so $p|_{V_j}$ is surjective.

Now, it suffices to show that $p|_{\overline{V}_j} \to \overline{U}$ is a homeomorphism. $p|_{\overline{V}_j}$ is bijective for the same reasons $p|_{V_j}$ is. Also, if $C \subseteq \overline{V}_j$ is closed, then, since $\overline{V}_j$ is compact, C is compact, so it's image under $p|_{\overline{V}_j}$ is compact and thus, since $\overline{U}$ is Hausdorff, closed. Hence $p|_{\overline{V}_j} : \overline{V}_j \to \overline{U}$ is a closed, bijective map and, therefore, a homeomorphism. Therefore, $p|_{V_j} : V_j \to U$ is a homeomorphism and so we see that U is evenly covered by p.

A similar argument demonstrates that the subset $U' \subseteq S^1$ consisting of all points on the circle having at least one negative coordinate is evenly covered by p. Every element in S^1 is contained in either U or U' (or both), so every element as a neighborhood evenly covered by p, to p is a covering map. $\square$

(b) What are all the non-trivial subgroups of $\mathbb{Z}$?

Answer: The non-trivial subgroups of $\mathbb{Z}$ are those groups of the form

$$n\mathbb{Z} = \{na : a \in \mathbb{Z}\}$$

for some $n \in \mathbb{Z}$.

To see this, let G be a subgroup of $\mathbb{Z}$. Since $\mathbb{Z}$ is well-ordered, there exists some positive, smallest $g \in G$. Let $x \in G$, $x \neq 0$. Then, by the Euclidean Algorithm, there exist $q, r \in \mathbb{Z}$ such that

$$x = qg + r$$

where $0 \leq r < g$. Now,

$$qg = \underbrace{g + g + \ldots + g}_{q \text{ summands}}$$

so $qg \in G$. Hence, since $x \in G$, $r = x - qg \in G$. Since g is the minimal positive element of G, this means it must be the case that $r = 0$. Therefore, every non-zero element of G is a multiple of g, so $G = g\mathbb{Z}$.

♣

(c) What subgroup of $\mathbb{Z}$ is $p_{n*}(\pi_n(S^1;1))$ isomorphic to?

Answer: Since $\mathbb{Z}$ and all of its subgroups are cyclic, this isomorphism will be completely determined by its action on the generator of $\pi_1(S^1;1)$, namely $[\sigma]$ where $\sigma : [0,1] \to S^1$ is given by $\sigma(t) = e^{i2\pi t}$. Now,

$$p_{n*}([\sigma]) = [p_n \circ \sigma]$$

where

$$(p_n \circ \sigma)(t) = p_n(\sigma(t)) = p_n(e^{i2\pi t}) = (e^{i2\pi t})^n = e^{i2\pi nt}.$$

If $\phi : \pi_1(S^1;1) \to \mathbb{Z}$ is the natural isomorphism, then $\phi(e^{i2\pi nt}) = n$. In other words, $p_{n*}([\sigma])$ corresponds to the element $n \in \mathbb{Z}$. Hence, $p_{n*}(\pi_1(S^1;1))$ is isomorphic to $n\mathbb{Z}$.

♣

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