Imaging Frequency-Dependent Reflectivity from Synthetic-Aperture Radar

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Abstract. This paper develops a method for using a synthetic-aperture radar system to obtain not only a spatial image of a scene but also the localized frequency dependence of the scene reflectivity. In other words, for each image pixel, we also obtain a plot of the frequency dependence of the reflectivity in that pixel. We present a method for extracting this information from the data, and also a formula that characterizes the performance of this imaging system. We conclude with some simulations suggesting that the method may be promising.

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1. Introduction

In synthetic-aperture radar (SAR) imaging, a moving antenna transmits electromagnetic pulses and receives the reflected energy, and from this data, images are formed by a variety of techniques [4, 12, 13, 14, 15, 19, 30], many of which are related to filtered backprojection.

For some remote sensing applications, it would be desirable not only to form an image of an object (target), but also to obtain information about the frequency dependence of the target’s radar reflectivity, which could be an indication of target material properties [1, 16, 23], such as water content or conductivity, or of the local geometry [24]. One of the challenges is that the frequency dependence constitutes another dimension, so that even for a two-dimensional target, the desired spatial-and-frequency image is a three-dimensional image, whereas SAR data normally depends on only two variables (antenna position and time delay of the pulse).
A clue to obtaining three-dimensional information from two-dimensional data is provided by circular SAR, in which height information can be obtained from a sequence of sub-aperture images [27]. This suggests that using a wide aperture might be important.

One approach [29, 33] to obtaining both frequency and spatial information is simply to divide the radar frequency band into sub-bands, and form images from the individual bands. These images are necessarily of lower resolution, but taken together, they provide information about the frequency dependence of the target radar reflectivity. This approach does not answer the question of whether there might be a better approach for combining the data to obtain the same information.

Some related work [32], that uses canonical correlation analysis to obtain spatially varying frequency information, has appeared in the sonar literature.

Another approach was developed in [5], which made the unrealistic assumption that the radar antenna could make multiple passes that effectively sweep out a two-dimensional surface of antenna positions, thus providing three-dimensional data.

A different approach is investigated in this paper. Conventional SAR systems transmit a sequence of identical pulses that each provide good range resolution. For such systems, the antenna can be approximated as stationary during the time that the pulse interacts with the scene; this is known as the start-stop or stop-and-shoot approximation. On the other hand, if a moving radar system instead transmitted waveforms with good resolution in both range and Doppler, it could potentially collect data depending on three variables, namely time delay, Doppler shift, and position. This paper develops an approach for using these three attributes to form a three-dimensional spatial-plus-frequency image.

From the radar uncertainty principle [11, 22] we know that there is a tradeoff in resolution between time delay and Doppler shift: both cannot be measured simultaneously with arbitrary accuracy. Consequently we expect a similar tradeoff in resolution between space and frequency.

Theory for SAR imaging without the start-stop approximation has been previously considered: the paper [31] analyzes the case of a stationary scene and chirp waveforms; [6] and [8] develop the theory for arbitrary waveforms and a scene consisting of multiple moving targets. In this paper we consider the case in which data from a single flight pass of a single antenna is used both to form an image and also to obtain information about the localized frequency dependence of regions of a flat surface.
2. The Mathematical model

2.1. Reduction to a scalar model

We consider the case in which the region between the sensors and the scattering objects consists of a homogeneous, lossless, non-dispersive atmosphere. In this situation, Maxwell’s equations can be used [18] to obtain an inhomogeneous wave equation for the electric field \( \mathcal{E} \):

\[
\nabla^2 \mathcal{E}(t, \mathbf{x}) - \frac{1}{c^2} \frac{\partial^2 \mathcal{E}(t, \mathbf{x})}{\partial t^2} = s(t, \mathbf{x})
\]  

(1)

and a similar equation for the magnetic field \( \mathcal{B} \). Here \( c \) denotes the speed of propagation of the wave in vacuum (which, for most remote sensing situations, is a good approximation to the speed in the atmosphere) and \( s \) is a source term that, in general, can involve \( \mathcal{E} \) and \( \mathcal{B} \). This source term is discussed in more detail below.

For simplicity we consider only one Cartesian component of equation (1):

\[
\left( \nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) \mathcal{E}(t, \mathbf{x}) = s(t, \mathbf{x}).
\]  

(2)

It will be useful to convert time-domain quantities to the frequency domain by means of the Fourier transform:

\[
\mathcal{E}(t, \mathbf{x}) = \frac{1}{2\pi} \int e^{-i\omega t} E(\omega, \mathbf{x}) d\omega.
\]  

(3)

In the frequency domain, the equation corresponding to (1) is the inhomogeneous Helmholtz equation:

\[
(\nabla^2 + k^2) E = S,
\]  

(4)

where the wave number \( k \) is defined as \( k = \omega/c \) and \( S \) is the Fourier transform of \( s \).

2.2. Scattering theory

2.2.1. Basic facts about the Helmholtz equation. The outgoing Green’s function [10] for the Helmholtz equation is

\[
G(\omega, \mathbf{x}) = \frac{e^{ik|x|}}{4\pi|x|}; \quad \text{it satisfies} \quad (\nabla^2 + k^2)G(\omega, \mathbf{x}) = -\delta(\mathbf{x}).
\]  

(5)
The Green’s function (5) enables us to solve the Helmholtz equation with an arbitrary source term on the right-hand side: in particular, the outgoing solution of
\[
(\nabla^2 + k^2)U(\omega, \mathbf{x}) = S(\omega, \mathbf{x}),
\]
is
\[
U(\omega, \mathbf{x}) = -\int G(\omega, \mathbf{x} - \mathbf{y})S(\omega, \mathbf{y})d\mathbf{y}.
\]

2.2.2. The Lippmann-Schwinger integral equation. In radar problems, the source \( S \) is a sum [9] of two terms, \( S = S^{\text{in}} + S^{\text{sc}} \), where \( S^{\text{in}} \) models the transmitting antenna, and \( S^{\text{sc}} \) models the scattering object [9]. The solution \( E \), which we write as \( E^{\text{tot}} \), therefore splits into two parts: \( E^{\text{tot}} = E^{\text{in}} + E^{\text{sc}} \). The first term, \( E^{\text{in}} \), which we call the incident field, satisfies the wave equation for the known, prescribed source \( S^{\text{in}} \). The second term \( E^{\text{sc}} \) is due to target scattering, and is called the scattered field. In our simplified scalar model, we write \( E^{\text{tot}} = E^{\text{in}} + E^{\text{sc}} \), where \( E^{\text{in}} \) satisfies the wave equation for the known, prescribed source \( S^{\text{in}} \).

In scattering problems the source term \( S^{\text{sc}} \) (typically) represents the target’s response to an incident field. This part of the source function will generally depend on the geometric and material properties of the target and on the form and strength of the incident field. Consequently, \( S^{\text{sc}} \) can be quite complicated to describe analytically, and in general it will not have the same direction as \( S^{\text{in}} \). Fortunately, for our purposes it is not necessary to provide a detailed analysis of the target’s response; for non-moving objects consisting of linear materials, we can write our scalar model \( S^{\text{sc}} \) as
\[
S^{\text{sc}}(\omega, \mathbf{x}) = -V(\omega, \mathbf{x})E^{\text{tot}}(\omega, \mathbf{x}),
\]
where \( V \) is the reflectivity function. It can display a sensitive dependence on \( \omega \) [17, 18, 26]. In the full vector case, \( V \) would be a matrix operating on the full vector \( E^{\text{tot}} \); here we use only the single matrix element corresponding to the component considered in (2).

We can use (8) in (7) to express \( E^{\text{sc}} \) in terms of the Lippmann-Schwinger integral equation [25]
\[
E^{\text{sc}}(\omega, \mathbf{x}) = \int G(\omega, \mathbf{x} - \mathbf{y})V(\omega, \mathbf{y})E^{\text{tot}}(\omega, \mathbf{y})d\mathbf{y}.
\]
2.2.3. The Born approximation. For radar imaging, we measure $E_{\text{sc}}$ at the antenna, and we would like to determine $V$ from these measurements. However, both $V$ and $E_{\text{sc}}$ in the neighborhood of the target are unknown, and in (9) these unknowns are multiplied together. This nonlinearity makes it difficult to solve for $V$. Consequently, almost all work on radar imaging involves making the Born approximation, which is also known as the weak-scattering or single-scattering approximation [20, 25]. The Born approximation replaces $E_{\text{tot}}$ on the right side of (9) by $E_{\text{in}}$, which is known.

$$E_{\text{sc}}(\omega, x) = \int \frac{e^{ik|x-z|}}{4\pi|x - z|} V(\omega, z) E_{\text{in}}(\omega, z) dz.$$  \hspace{1cm} (10)

The Born approximation is very useful, because it makes the imaging problem linear. It is not, however, always a good approximation [7], particularly when multiple scattering effects are important.

2.3. The field transmitted from a moving antenna

We model the antenna as an isotropic point source located at $\gamma(t)$ (in some fixed reference frame), transmitting the waveform $f(t)$. The corresponding equation for the time-domain field emanating from the antenna is then

$$\left(\nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2}\right) E_{\text{in}}(t, x) = -f(t)\delta(x - \gamma(t)).$$  \hspace{1cm} (11)

In the frequency domain, (11) is

$$(\nabla^2 + k^2) E_{\text{in}}(\omega, x) = -\int e^{i\omega t} f(t) \delta(x - \gamma(t)) dt,$$  \hspace{1cm} (12)

and consequently the incident field, i.e., the field radiating from the antenna, is [21, 6]

$$E_{\text{in}}(\omega, x) = \int e^{ik|x-y|} \int e^{i\omega t} f(t) \delta(y - \gamma(t)) dt dy = \int e^{i\omega [t + |x - \gamma(t)|/c]} \frac{1}{4\pi|x - \gamma(t)|} f(t) dt.$$  \hspace{1cm} (13)

Note that the signs in the phase of (13) seem odd because the integration is over $t$ rather than the more usual $\omega$. 

If we introduce the notation \( R_x(t) = x - \gamma(t) \) and \( R_x(t) = |R_x(t)| \) and write \( f \) in terms of its Fourier transform as
\[
f(t) = \frac{1}{2\pi} \int e^{-i\omega't} F(\omega')d\omega',
\]
then (13) is
\[
E^{\text{inc}}(\omega, x) = \int \frac{e^{i\omega[t + R_x(t)/c]}}{4\pi R_x(t)} \frac{1}{2\pi} \int e^{-i\omega't} F(\omega')d\omega'dt.
\]

Remark: The transmitted wave is Doppler shifted. The stationary phase theorem shows\( \dagger \) that the leading-order contribution to (15) comes from the set
\[
0 = \frac{\partial}{\partial t} \left( \omega[t + R_x(t)/c] - \omega't \right) = \omega \left[ 1 - \frac{\dot{R}_x(t)}{c} \right] - \omega'.
\]
Equation (16) shows that the main contribution to the radiated field \( E^{\text{inc}}(\omega, x) \) is at frequency
\[
\omega = \frac{\omega'}{1 - \dot{R}_x(t)/c} = \omega' \left( 1 + \frac{\dot{R}_x(t)}{c} + \cdots \right).
\]
In other words, in the reference frame in which the path is \( \gamma \), transmitted frequencies are Doppler-shifted.

2.4. The scattered field

In the Born scattering model (10), we use expression (13) for the incident field:
\[
E^{sc}_B(\omega, x) = \int \frac{e^{i\omega|x-y|/c}}{4\pi|x-y|} V(\omega, y) E^{\text{inc}}(\omega, y) dy
= \int \frac{e^{i\omega|x-y|/c}}{4\pi|x-y|} V(\omega, y) \int \frac{e^{i\omega[t' + R_y(t')/c_0]}}{4\pi R_y(t')} f(t')dt' dy.
\]
where we have used (13). Note that because above we have shown that the leading-order contributions to the incident field come from Doppler-shifted frequencies, the interaction with the target \( V \) takes place at the Doppler-shifted frequency.

\( \dagger \) For flight paths and target locations satisfying \( \partial^2_t R_x(t) = 0 \), the critical point is degenerate and the analysis is more complicated.
2.5. The signal received on a moving antenna

Because we measure the scattered field at a moving sensor, we transform (18) into the time domain to obtain

\[ E_{sc}^B(t, \mathbf{x}) = \frac{1}{2\pi} \int e^{-i\omega t} E_{sc}^B(\omega, \mathbf{x}) d\omega \]

\[ = \frac{1}{2\pi} \int e^{-i\omega t} \int \frac{e^{i\omega|x-y|/c}}{4\pi|x-y|} V(\omega, y) \int \frac{e^{i\omega[t'+R_y(t')/c]}}{4\pi R_y(t')} f(t') dt' dy d\omega. \quad (19) \]

The received signal on the moving sensor (see [21]) is then

\[ E_{sc}^B(t, \gamma(t)) = \frac{1}{2\pi} \int e^{-i\omega t} \int \frac{e^{i\omega R_y(t)/c}}{4\pi R_y(t)} V(\omega, y) \int \frac{e^{i\omega[t'+R_y(t')/c]}}{4\pi R_y(t')} f(t') dt' dy d\omega \]

\[ = \frac{1}{2\pi} \int \frac{e^{-i\omega[t-t'-R_y(t)/c-R_y(t')/c]}}{(4\pi)^2 R_y(t) R_y(t')} f(t') dt' V(\omega, y) d\omega dy. \quad (20) \]

Stationary phase analysis can be used to verify that the received signal is Doppler shifted; details can be found in the Appendix.

Note that the parametrization of \( V \) involves a choice of reference frame; it is this reference frame in which the antenna path is \( \gamma \). Here we are assuming that time is the same in all reference frames. If we were to include corrections from special relativity, then we should transform to the sensor proper time to determine the transmitted and received signals. However, the corrections of special relativity are of order \((|\dot{\gamma}|/c)^2\) and are assumed here to be negligible.

2.6. Signals from a train of pulses.

We consider the case in which the antenna illuminates only a limited region of the earth.

We take \( f \) to be a train of pulses of the form

\[ f(t) = \sum_m f_m(t - T_m) \quad m = 0, 1, 2, \ldots \quad (21) \]

where the pulse \( f_m \) is assumed to be supported in the interval \([-\delta_m, \delta_m]\) and where the delay between successive pulses is sufficiently large so that returns from successive pulses do not overlap.
Then the time-domain data from the \( m \)th pulse is, from (20),
\[
d(t, m) \propto \int \frac{e^{-i\omega[t-t'-R_y(t)/c-R_y(t')/c]}f_m(t' - T_m)dt'}{(4\pi)^2 R_y(t)R_y(t')} \cdot V(\omega, \mathbf{y}) \, d\omega d\mathbf{y}.
\]  
(22)

### 2.7. Data under the slow-mover approximation

If the antenna is not moving too fast, then for each pulse \( m \) we can expand \( R_y(t') = |\mathbf{y} - \gamma(t')| \) (the distance from the transmitter to the target at \( \mathbf{y} \)) around \( T_m \):
\[
R_y(t') = |\mathbf{y} - \gamma_m(t' - T_m) + \cdots| = R_{m, \mathbf{y}} - \dot{R}_{m, \mathbf{y}}(t' - T_m) + \cdots,
\]  
(23)
where \( \gamma_m = \gamma(T_m), R_{m, \mathbf{y}} = |\mathbf{y} - \gamma_m|, R_{m, \mathbf{y}} = |R_{m, \mathbf{y}}|, \) and \( \dot{R}_{m, \mathbf{y}} = \hat{R}_{m, \mathbf{y}} - \gamma_m \), where the hat denotes unit vector: \( \hat{\mathbf{a}} = \mathbf{a}/|\mathbf{a}| \). If we assume that \( \dot{\gamma}/c \ll 1 \), then with the change of variables \( \tau' = t' - T_m \), we can write the data (22) from the \( m \)th pulse as
\[
d(t, m) \propto \int \frac{e^{-i\omega[t-t'-T_m-|\gamma(t)-\mathbf{y}|/c-R_{m, \mathbf{y}}/c+R_{m, \mathbf{y}}\tau'/c+\cdots]}f_m(\tau')d\tau'}{(4\pi)^2 |\gamma(t)-\mathbf{y}| R_{m, \mathbf{y}}} \cdot V(\omega, \mathbf{y}) \, d\omega d\mathbf{y}.
\]  
(24)

If the pulses are short and the antenna is not too distant from the target scene, then in (24) we can also expand \( R_y(t) = |\mathbf{y} - \gamma(t)| \) (the distance from the target at \( \mathbf{y} \) to the receiver) about \( t = T_m \):
\[
|\gamma(t) - \mathbf{y}| = |\gamma_m + \gamma_m(t - T_m) + \cdots - \mathbf{y}| = R_{m, \mathbf{y}} - \dot{R}_{m, \mathbf{y}}(t - T_m) + \cdots
\]  
(25)

Using this expansion in (24) and writing \( \tau = t - T_m \), we obtain
\[
d(\tau, m) \propto \int \frac{e^{-i\omega[\tau-\gamma_m(t)/c+R_{m, \mathbf{y}}\tau'/c+\cdots]}f_m(\tau')d\tau'}{(4\pi)^2 R_{m, \mathbf{y}}} \cdot V(\omega, \mathbf{y}) \, d\omega d\mathbf{y}
\]
\[
= \int \frac{e^{-i\omega[\tau-2R_{m, \mathbf{y}}/c+R_{m, \mathbf{y}}(\tau+\tau')/c+\cdots]}f_m(\tau')d\tau'}{(4\pi)^2 R_{m, \mathbf{y}}} \cdot V(\omega, \mathbf{y}) \, d\omega d\mathbf{y}
\]
\[
= \int \frac{e^{-i\omega[\tau-2\beta_{m, \mathbf{y}}/c+\beta_{m, \mathbf{y}}(\tau+\cdots]}f_m(\omega(1 - \beta_{m, \mathbf{y}})) \cdot V(\omega, \mathbf{y}) \, d\omega d\mathbf{y}
\]
\[
\approx \int \frac{e^{-i\omega[\tau+\beta_{m, \mathbf{y}}-2R_{m, \mathbf{y}}/c]}f_m(\omega(1 - \beta_{m, \mathbf{y}})) \cdot V(\omega, \mathbf{y}) \, d\omega d\mathbf{y}.
\]  
(26)
where $\beta_{m,y} = \hat{R}_{m,y} \cdot \gamma_m/c = \hat{R}_{m,y}/c$.

In expanding distances $R_y(t)$ around $t = T_m$ and $R_y(t')$ around $t' = T_m$ in the expressions for each pulse, we are using approximations that are good for $t$ and $t'$ close to $T_m$. In other words, we are assuming that the sensor is relatively close to the target, so that the propagation time delay does not introduce large errors into the expressions for the distances. For some satellite systems, this might not be the case; for such systems, the more detailed analysis of [8] can be incorporated.

Equation (26) is our model for the received signal from the $m$th pulse. We see that it incorporates the Doppler shift due to transmission from the moving antenna, interaction with the target at the Doppler-shifted frequency, the Doppler shift due to reception on the moving antenna, and the two-way travel time from the antenna to the target and back.

3. Image formation

We form an image $I(\omega', z)$ of $V(\omega', z)$ by weighting and matched filtering the data at each pulse, and then summing over the pulses:

$$I(\omega', z) \propto \sum_m R_{m,z}^2 F_m^*(\omega'(1 - \beta_{m,z})) \int e^{i\omega'[\tau(1 + \beta_{m,z}) - 2R_{m,z}/c]} d(\tau, m) d\tau.$$  (27)

where $*$ denotes complex conjugate.

4. Image analysis

4.1. The point-spread function

To determine how the image $I(\omega', z)$ formed via (27) is related to the true reflectivity $V(\omega, y)$, we substitute the data model (26) into the formula (27) for image formation:

$$I(\omega', z) \propto \sum_m R_{m,z}^2 \int e^{i\omega'[\tau - 2R_{m,z}/c + \beta_{m,z}\tau + \cdots]} F_m^*(\omega'(1 - \beta_{m,z}))$$

$$\times \int e^{-i\omega'[\tau - 2R_{m,y}/c + \beta_{m,y}\tau + \cdots]} F_m(\omega(1 - \beta_{m,y})) d\tau V(\omega, y) d\omega dy.$$  (28)

We interchange the order of integration; the result can be written

$$I(\omega', z) = \int K(\omega', z; \omega, y)V(\omega, y)d\omega dy,$$  (29)
where

\[ K(\omega', z; \omega, y) \propto \sum_m \int e^{i\omega'[\tau - 2R_{m,z}/c + \beta_{m,z}\tau + ...]} F_m^*(\omega'(1 - \beta_{m,z})) \frac{R_{m,z}^2}{R_{m,y}^2} \times \int e^{-i\omega[\tau - 2R_{m,y}/c + \beta_{m,y}\tau + ...]} F_m(\omega(1 - \beta_{m,y})) \, d\tau \]

\[ = \sum_m \frac{R_{m,z}^2}{R_{m,y}^2} \int e^{i\tau(\omega'[1 + \beta_{m,z}] - \omega[1 + \beta_{m,y}]}) \, d\tau \times e^{-2i\omega'R_{m,z}/c} e^{2i\omega'R_{m,y}/c} \]

\[ \times F_m^*(\omega'(1 - \beta_{m,z})) F_m(\omega(1 - \beta_{m,y})) \] (30)

is the point-spread function. If the point-spread function were a delta function then the image formed by (27) would be a perfect one.

The \( m \)th pulse \( f_m(t - T_m) \) is supported in the interval \([T_m - \delta_m, T_m + \delta_m]\); consequently we can write the \( \tau \) integral of (30) as

\[ \int_{\delta_m} \delta_m e^{i\tau(\omega'[1 + \beta_{m,z}] - \omega[1 + \beta_{m,y}]}) \, d\tau = 2\delta_m \text{sinc} \left[ \delta_m(\omega'[1 + \beta_{m,z}] - \omega[1 + \beta_{m,y}]) \right]. \] (31)

where \( \text{sinc} \theta = (\sin \theta)/\theta \). Then, using (31) in (30), we obtain the expression

\[ K(\omega', z; \omega, y) \propto \sum_m \frac{R_{m,z}^2}{R_{m,y}^2} \delta_m e^{2i(\omega R_{m,y} - \omega'R_{m,z})/c} F_m^*(\omega'(1 - \beta_{m,z})) F_m(\omega(1 - \beta_{m,y})) \]

\[ \times \text{sinc} [\delta_m(\omega'[1 + \beta_{m,z}] - \omega[1 + \beta_{m,y}])]. \] (32)

4.2. Analysis of the PSF

It does not appear that the PSF (30) or (32) can be written in the usual way [7] as the kernel of a pseudodifferential operator, and consequently its behavior cannot be analyzed by the usual approach. Instead we consider a number of special cases and show some plots for a numerical example.

4.2.1. Extreme narrowband waveform. If the transmitted waveform is \( f(t) = e^{-i\omega_0 t} \), with \( F(\omega) \propto \delta(\omega - \omega_0) \), and the sinc function of (32) is approximated as a delta function, then (30) leads to the equations

\[ \omega_0 = \omega'(1 - \beta_{m,z}) \]

\[ \omega_0 = \omega(1 - \beta_{m,y}) \]

\[ \omega'[1 + \beta_{m,z}] = \omega[1 + \beta_{m,y}]. \] (33)
When the first two lines of (33) are solved for \( \omega' \) and \( \omega \), respectively, and the results substituted into the third line, we obtain

\[
\omega_0 \frac{1 + \beta_{m,z}}{1 - \beta_{m,z}} = \omega_0 \frac{1 + \beta_{m,y}}{1 - \beta_{m,y}},
\]

which implies that \( \beta_{m,z} = \beta_{m,y} \), or, from the definition of \( \beta \) below (26), \( \hat{R}_{m,z} = \hat{R}_{m,y} \).

This condition says that at each look \( m \), the antenna velocity in the direction of \( z \) must be the same as the antenna velocity in the direction of \( y \). Using \( \beta_{m,z} = \beta_{m,y} \) in the first two lines of (33) again, we see that \( \omega = \omega' \). We know from [2] that spatial image formation can be done by combining Doppler information from a sequence of slow-time views. Mathematically, this can be seen by approximating the sum over \( m \) in (30) by an integral and applying the method of stationary phase. Differentiating with respect to \( m \) gives an \( \ddot{R} \) equation that, combined with the \( \dot{R} \) (Doppler) information, locates points spatially. Curves of constant \( \dot{R} \) and constant \( \ddot{R} \) are shown in Figure 1 for an antenna at the origin; clearly these two families of curves provide a coordinate system similar to the \( R \) and \( \dot{R} \) coordinate system that arises in standard SAR [7].

**Figure 1.** Curves of constant \( \dot{R} \) and constant \( \ddot{R} \) for an antenna at the origin moving along the \( y \) axis.
If only a single frequency $\omega_0$ is transmitted, then the target is interrogated only by waves whose frequencies are the Doppler-shifted versions of $\omega_0$. This limits the range of target frequencies that can be assessed.

Technically, of course, a pulsed system must transmit more than a single frequency; however the above analysis provides some insight into what would be expected from a system in which each pulse consists of a “CW (continuous wave) pulse”, i.e., a single long “tone”.

4.2.2. Extreme wideband waveform. If $f(t) = \delta(t)$ with $F(\omega) = 1$, then from the method of stationary phase applied to (30), we obtain

\[ \omega'[1 + \beta_{m,z}] = \omega[1 + \beta_{m,y}] \]

\[ \omega'\beta_{m,z} = \omega\beta_{m,y}, \]

(35)

where the first equation comes from differentiating the phase with respect to $\tau$ and the second equation comes from “differentiating” the phase of the exponentials with respect to $m$. Using the second line of (35) in the first produces the result $\omega' = \omega$; this in turn implies that $\beta_{m,z} = \beta_{m,y}$, or in other words, $\hat{R}_{m,z} = \hat{R}_{m,y}$. Thus again the spatial image formation is accomplished by the Doppler SAR reconstruction of [2].

Unfortunately it is difficult to compare the resolution from (32) with that of standard SAR, because a spatial slice of the PSF corresponds to a target that scatters only at the frequency $\omega = \omega_0$. Standard SAR, on the other hand, assumes that the target scatters all frequencies equally, and SAR imaging algorithms combine these frequencies to obtain range resolution.

4.2.3. Target with gentle variation in frequency. If the reflectivity function $V(\omega, y)$ varies slowly in $\omega$, then $V$ can be approximated (for example by linear interpolation) from its values at a few frequencies. In this case, the transmitted frequency band can be split up into intervals around these frequencies, and separate images formed from each band. If, in addition, the $\beta$ (Doppler terms) are negligible, then (29) and (30) can be written

\[ I(\omega^*, z) = \int K_{\omega^*}(z, y)V(\omega^*, y)dy, \]

(36)
where
\[ K_{\omega^*}(z, y) \propto \sum_m \frac{R_{m,z}^2}{R_{m,y}^2} \int e^{-2\omega(R_{m,z} - R_{m,y})/c} |F_m(\omega)|^2 d\omega. \] (37)

where the \( \omega \) integration is over a sub-band around \( \omega^* \). This is the standard SAR point-spread function for the sub-band image. It is clear that this approach for extracting frequency information provides a spatial image with coarser range resolution than could be obtained from using all the frequencies to form one spatial image.

4.3. Numerical example

We show plots of the PSF (32) for the case of a waveform with \( F_m \equiv 1 \) for the relevant frequencies. We assume that the system transmits 200 identical pulses, each of duration .009 seconds, transmitted one per second, with a center frequency of 1 GHz. Thus each pulse has a half bandwidth of about \( \pi/\text{duration} \approx 350 \text{ Hz} \). In other words, each time-domain pulse is approximately given by
\[ f_m(t) \approx \int_{2\pi(10^9-350)}^{2\pi(10^9+350)} e^{-i\omega t} d\omega. \] (38)

We use a straight flight path in the \( y_2 \) direction with a height of 10 km, a 45\(^\circ\) angle of elevation, and a speed of 50 m/sec. The 200-second coherent processing interval multiplied by the platform speed of 50m/sec gives a synthetic aperture of 10 km, or about 40 degrees.

We note that with this geometry, the round-trip travel time is about \( 10^{-4} \) sec. During that time, the antenna moves about \( 5 \cdot 10^{-3} \) m, which is much smaller than the wavelength, and consequently the assumptions discussed below (26) are justified.

We show the plot of \( |K(\omega', z; 2\pi 10^9, 0)| \); this is the same as the (magnitude of the) image for \( V(\omega, y) = \delta(\omega - \omega_0)\delta(y) \) where \( \omega_0 = 2\pi f_0 \) with \( f_0 = 1 \) GHz. In other words, the target is assumed to scatter only at the frequency 1 GHz.

Figures 2 and 3 show the spatial slice at \( \omega' = \omega_0 \). Figure 3 is simply an enlarged version of Figure 2. Here both axes (\( z_1 \) and \( z_2 \)) are in meters. From Figure 3, we can estimate the cross-range resolution at a few meters. The cross-range resolution is degraded with a narrower synthetic aperture, as expected. The null-to-null cross-range resolution for a standard SAR system with the same synthetic aperture used
Figure 2. A plot of $|K(2\pi 10^9, z; 2\pi 10^9, 0)|$. The labels on both axes are in meters.

for Figure 3 and center frequency of 1 GHz would be about half a meter. Thus we see, as expected, that extracting frequency information involves relinquishing spatial resolution.

Figure 4 shows the PSF at the correct frequency ($|K(2\pi 10^9, z; 2\pi 10^9, 0)|$ again, the same as Figure 2 ) and a slice at 100 Hz above the correct frequency ($|K(2\pi (10^9 + 100), z; 2\pi 10^9, 0)|$), both with approximately the same color scale. Other constant-frequency slices away from $f_0$ give images that are similarly close to zero.

Figure 5 shows the slice through $z = 0$; this gives the (magnitude of the) target frequency dependence. The frequency resolution appears to be about a hundred Hertz. A longer-duration waveform results in a narrower peak, as expected. A narrower angular aperture results in a broader peak, which means that the frequency resolution is improved for a flight path providing a wide aperture. This is consistent with the expectation that more information can be obtained from a wider angular aperture.
5. Discussion and future work

We have developed a method for using a SAR system for obtaining the spatially varying, frequency-dependent reflectivity of a scene. The theory can handle any flight trajectory and any sequence of transmitted waveforms; the SAR receiver is assumed to be able to measure instantaneous Doppler shifts as well as time delays. The theory provides a formula for the image formation process, and a formula for the point-spread function of the imaging system; consequently the theory provides a tool for system design. The simulations suggest that this approach may be promising, particularly for wide-angle systems.

An alternative approach can be pursued, in which the frequency dependence of the reflectivity is expressed as a time-domain convolution. From this approach, the challenge is to disentangle the signal delay due to range from the signal delay due to the material interaction. These two delays, however, depend differently on the antenna position. This time-domain approach does allow the corresponding PSF
Figure 4. Top: $|K(2\pi 10^9, z; 2\pi 10^9, 0)|$ again (same as Figure 3 but showing a larger spatial region). Bottom: $|K(2\pi (10^9 + 100), z; 2\pi 10^9, 0)|$ with the same color scale.
to be written as a pseudodifferential operator. However, we have chosen to explore the present frequency-domain formulation because of its natural connection with the sub-band imaging approach and with the frequency information sought.

We leave for the future a more complete exploration of the properties of the PSF for different sequences of waveforms and different flight paths.

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7. References


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Appendix A.

The received signal is Doppler-shifted. The frequency-domain signal model is obtained by Fourier transforming (20):

\[
D(\omega) = \int e^{i\omega t} \mathcal{E}_B(t, \gamma(t)) dt
\]

\[
\propto \int e^{i\omega t} \int \frac{e^{-i\omega'[t-t' - \frac{R_y(t)}{c} - \frac{R_y(t')}{c}]}}{(4\pi)^2 R_y(t) R_y(t')} f(t') dt' V(\omega', y) d\omega'dy dt. \tag{A.1}
\]

Stationary phase analysis in the \( t \) and \( \omega' \) variables shows that the leading-order contribution \( \ddagger \) comes from the values of \( t \) and \( \omega' \) satisfying the critical conditions for the phase \( \phi = \omega t - \omega'[t - t' - \frac{R_y(t)}{c} - \frac{R_y(t')}{c}] \), namely

\[
0 = \frac{\partial \phi}{\partial t} = \omega - \omega'(1 - \frac{\dot{R}_y(t)}{c})
\]

\[
0 = \frac{\partial \phi}{\partial \omega'} = t - t' - \frac{R_y(t)}{c} - \frac{R_y(t')}{c}, \tag{A.2}
\]

where \( \dot{R}_y(t) = \dot{\hat{R}}_y(t) \cdot \dot{\gamma}(t) \) with \( \hat{R}_y = \frac{R_y}{R_y} \). Thus the leading-order contribution to \( D \) is proportional to

\[
D(\omega) \propto \int \frac{f(t')}{(4\pi)^2 R_y(t) R_y(t')} V\left(\frac{\omega}{1 - \frac{R_y(t)}{c}}, y\right) \bigg|_{t=t^*} dt' dy, \tag{A.3}
\]

where \( t^* \) satisfies the second condition of (A.2). We see that the received field depends on the reflectivity at a Doppler-shifted frequency.

\( \ddagger \) Note that the Hessian is never zero because \( \dot{R}_y < c \)