Microlocal Structure of Inverse Synthetic Aperture Radar Data

Margaret Cheney†‡ and Brett Borden†§

† Research Department, Naval Air Warfare Center Weapons Division, China Lake, CA 93555-6100 USA

‡ permanent address: Department of Mathematical Sciences, Rensselaer Polytechnic Institute, Troy, NY 12180 USA

§ permanent address: Physics Department, Naval Postgraduate School, Monterey, CA 93943 USA

Abstract. We consider the problem of all-weather identification of airborne targets. We show that structural elements of the target correspond to identifiable features of the radar data. Our approach is based on high-frequency scattering methods but is not limited to the standard weak-scatterer approximation: we also analyze multiple scattering and structural dispersion (situations normally interpreted in terms of poorly-behaved image "artifacts"). This work suggests a method for target identification that circumvents the need to create an intermediate radar image from which the object's characteristics are to be extracted. As such, this scheme may be applicable to efficient machine-based radar identification programs.

PACS numbers: 41.20.Jb, 42.30.Wb

1. Introduction

Object identification from reflected radio waves is an inverse problem with a long history. This challenging problem is still mostly unsolved but the impetus for the work is high because, if perfected, such methods would allow for reliable recognition of non-cooperating *targets* in all types of weather and at great distances. Radar-based target recognition efforts share a great deal in common with other problems of remote sensing and current practice attempts to perform target identification/classification from fully formed radar images. Of course, constructing an image of a target from radar data is a very difficult task all by itself since the reflected field data are noisy and are usually collected from a very limited set of (generally unknown) target orientations [19, 29]. Additional complications arise in the real-time implementation of imaging algorithms in realizable radar systems.

Automatic classification systems, however, should be able to skip this imaging step because a fully-formed image is probably not required for machine-based target recognition. This observation, of course, begs the question of "what components of the raw data set are relevant to target identification?" In this paper we examine a systematic method for extracting structure-relevant information directly from measured radar data without the need to first construct an image of the target.

Our approach relates the singular structure (such as edges) of the target to the singular structure of the data set. Restricting our attention to the singular structure—specifically, to a certain set in phase space called the *wavefront set*—allows us to use the tools of microlocal analysis [9, 13, 32]. This strategy was first applied to imaging problems in [1]; its uses in

seismic prospecting [2, 6, 10], X-ray tomography [11, 16], and Synthetic-Aperture Radar [23] are active areas of research. An approach similar to the one we pursue here, in which we use microlocal analysis not to do imaging but instead to study the connection between features of the target and the data, was considered for the X-ray tomography problem by Quinto [25].

We begin in section 2 by examining the general properties of radar scattering and developing mathematical models for the measured data. These models involve Fourier Integral Operators with kernels that are oscillatory integrals; consequently these models can be studied with the techniques of microlocal analysis. Next we present an overview of the microlocal concepts and theorems that are relevant to our investigation (section 3.1). These two sections serve to introduce our notation and define our terminology. Section 3 contains our main results and calculates the wavefront sets associated with several important radar scattering situations: weak scatterers; multiple scatterers; and structural dispersion (all cases are limited to targets whose rotational acceleration is negligible). We conclude our discussion in section 4 by considering several connections between our results and existing practices and problems of radar imaging systems. In particular, we briefly discuss targets whose behavior is not well modeled by our assumptions and suggest potentially fruitful paths for further research.

2. Radar data

Traditional radar systems transmit an electromagnetic waveform (a *pulse*) and measure the time delay and frequency shift of the corresponding waveform reflected from a target so as to estimate that target's range and speed. When very short-duration pulses are used, it is possible to accurately determine the range to individual target substructures. Such *high range resolution* (HRR) radar systems can be used to obtain a target's local integrated scattering strength as a function of its range.

These one-dimensional "images" are known as *range profiles* and are used by many all-weather target recognition systems. But target-identification procedures based on range profiles suffer from a lack of target information in dimensions orthogonal to range since range-only radar data maps the reflected energy from all equidistant target substructures to the same point. Such ambiguity can be partially removed by considering multiple pulses that interrogate the target from different directions. The different target views, which are also known as target *aspects*, collectively define a *synthetic aperture* and more complete target images can be recovered from multi-aspect data by, for example, backprojection methods. In principle, there are two basic schemes for creating synthetic apertures: either the radar measurement system can move relative to the target (a configuration known as synthetic-aperture radar, or SAR); or the target can rotate and sequentially present different aspects to the radar (a situation known as inverse synthetic-aperture radar, or ISAR). In practice, of course, one usually sees a combination of these idealized cases and the terminology is somewhat artificial.

Evidently, cross-range resolution depends on the size of the synthetic aperture. In ISAR systems, this means that cross-range resolution will also be related to the length of the time interval over which these data are collected because the observer must wait for an aperture to be established by the rotating target. For a well-behaved target (i.e., one rotating at constant rate), cross-range resolution therefore depends on the duration of the measurements. Rotational target motion also induces a differential Doppler shift in the target's cross-range dimension. This observation is the reason why ISAR imaging is sometimes called "range-Doppler imaging" (although, in HRR systems the Doppler shift associated with target rotation is usually too small to be directly measured on a pulse-by-pulse basis).

Ultimately, the behavior of radar data is determined by scattered-field solutions to the wave equation. Since radar systems transmit and receive radio waves, we should generally examine the electromagnetic (vector) wave equation. For simplicity, however, we will examine the scalar wave equation and assume that the components of the electromagnetic field each satisfy

$$\left(\nabla^2 - c^{-2}(t,\boldsymbol{x})\,\partial_t^2\right) u(t,\boldsymbol{x}) = 0\,. \tag{1}$$

We write the total field as a sum of the incident and scattered fields $u = u^{\mathrm{inc}} + u^{\mathrm{sc}}$; the resulting equation for u^{sc} is

$$\left(\nabla^2 - c^{-2}\,\partial_t^2\right) u^{\mathrm{sc}}(t,\boldsymbol{x}) = -V(t,\boldsymbol{x})\,\partial_t^2 u(t,\boldsymbol{x})\,, \tag{2}$$

where $V(t,\boldsymbol{x}) = c^{-2} - c^{-2}(t,\boldsymbol{x})$ is the target scattering density at time t and position $\boldsymbol{x} \in \mathbb{R}^3$. We can write (2) as an integral equation

$$u^{\mathrm{sc}}(t,\boldsymbol{x}) = \int g(t-t',\boldsymbol{x}-\boldsymbol{y})V(t',\boldsymbol{y})\,\partial_{t'}^2 u(t',\boldsymbol{y})\,\mathrm{d}t'\mathrm{d}\boldsymbol{y}\,, \tag{3}$$

where [31]

$$g(t,\boldsymbol{x}) = \frac{\delta(t-|\boldsymbol{x}|/c)}{4\pi|\boldsymbol{x}|} = \int \frac{\mathrm{e}^{-\mathrm{i}\omega(t-|\boldsymbol{x}|/c)}}{8\pi^2|\boldsymbol{x}|}\,\mathrm{d}\omega \tag{4}$$

satisfies $\left(\nabla^2 - \partial_t^2\right) g(t,\boldsymbol{x}) = -\delta(t)\delta(\boldsymbol{x})$.

In section 2.1, we develop a mathematical model for radar data and explain the fundamental role played by the weak scattering approximation. We examine the multiple-scattering case in section 2.2, where we construct an exact scattering solution for two isotropic point scatterers. In section 2.3, we consider a model for scattering from a reentrant structure such as a duct or engine inlet.

2.1. Weak scattering

There are a variety of situations when the approximation known as the *Born approximation*, *single-scattering approximation*, or *weak-scattering approximation* is appropriate [15, 17]. Under this approximation, we replace the full field u on the right side of (2) and (3) by the incident field u^{inc}, which converts (3) into

$$u^{\mathrm{sc}}(t,\boldsymbol{x}) \approx \int g(t-t',\boldsymbol{x}-\boldsymbol{y})V(t',\boldsymbol{y})\,\partial_{t'}^2 u^{\mathrm{inc}}(t',\boldsymbol{y})\,\mathrm{d}t'\mathrm{d}\boldsymbol{y}\,. \tag{5}$$

The value of this approximation is that it removes the nonlinearity in the inverse problem: it replaces the product of two unknowns (V and u) by a single unknown (V) multiplied by the known incident field.

For radar measurement systems, the single scattering approximation is the basis for a crucial method for estimating the scattered field in the presence of system noise. This is a serious issue, because the energy of the scattered field at the receiver will be reduced by at least a factor of R^{-4} (where R denotes the distance between the radar and the target and typically ranges from ten to one hundred kilometers). Thus the signal measured by the radar will typically be small in comparison with the thermal noise voltage. This difficulty can be overcome by correlating the received signal with a model of the expected reflection signal; by this means, radar systems can significantly reduce the effects of system noise and extend the effective range R without having to increase the energy of the transmitted signal to impossible levels [4, 8]. The signal model generally used for such measurements is based on the single scattering approximation: the scattered field is presumed to be a time- and

frequency-shifted replica of $u^{\rm inc}$. The term "radar data" usually refers to these correlation-receiver measurements.

We assume that the incident field is a series of pulses, beginning at times $t = \theta_n, n = 1, 2, \ldots$ from an isotropic point radiator at position $\boldsymbol{x}$, so that

$$u_n^{\rm inc}(t', \boldsymbol{y}) = \int S_{\rm inc}(\omega') \frac{{\rm e}^{-{\rm i}\omega'(t'-\theta_n-|\boldsymbol{x}-\boldsymbol{y}|/c)}}{8\pi^2|\boldsymbol{x}-\boldsymbol{y}|}\, {\rm d}\omega' \,, \tag{6}$$

where

$$S_{\rm inc}(\omega) = \mathcal{F}\{s_{\rm inc}\}(\omega) = \frac{1}{2\pi}\int s_{\rm inc}(t')\, {\rm e}^{{\rm i}\omega t'}\, {\rm d}t' \tag{7}$$

is the Fourier transform of the signal used to establish the interrogating field transmitted to the target. We also assume that the target is translating with velocity $\boldsymbol{v}$ and rotating, so that at time t', we have $V(t', \boldsymbol{y}) = Q(\mathcal{O}^{-1}(t')(\boldsymbol{y} - \boldsymbol{v}t'))$, where $\mathcal{O}(t')$ denotes a rotation operator (an orthogonal matrix).

We consider the *monostatic* case, in which the transmitter and receiver are co-located. At the radar, the field due to the nth transmitted pulse is thus $u_n^{\rm sc}(t, \boldsymbol{x})$. This field induces a system signal whose Born-approximated value we denote by $s_{\rm sc}(\boldsymbol{x}, n, t)$:

$$\begin{aligned} s_{\rm sc}(\boldsymbol{x}, n, t) = \int & \frac{{\rm e}^{-{\rm i}\omega(t-t'-|\boldsymbol{x}-\boldsymbol{y}|/c)}}{8\pi^2|\boldsymbol{x}-\boldsymbol{y}|} Q_{\rm w}(\mathcal{O}^{-1}(t')(\boldsymbol{y}-\boldsymbol{v}t')) \\ & \times ({\rm i}\omega')^2 S_{\rm inc}(\omega') \frac{{\rm e}^{-{\rm i}\omega'(t'-\theta_n-|\boldsymbol{x}-\boldsymbol{y}|/c)}}{8\pi^2|\boldsymbol{x}-\boldsymbol{y}|}\, {\rm d}\omega' {\rm d}\omega {\rm d}t' {\rm d}\boldsymbol{y} \,. \end{aligned} \tag{8}$$

In (8), we neglect the overall target velocity (set $\boldsymbol{v} = 0$), let $t'' = t' - \theta_n$, and make the change of variables $\boldsymbol{z} = \mathcal{O}^{-1}(t'' + \theta_n)\boldsymbol{y}$. This approximation converts (8) into

$$\begin{aligned} s_{\rm sc}(\boldsymbol{x}, n, t) = \int & \frac{{\rm e}^{-{\rm i}\omega(t-t''-\theta_n-|\boldsymbol{x}-\mathcal{O}(t''+\theta_n)\boldsymbol{z}|/c)}}{8\pi^2|\boldsymbol{x}-\mathcal{O}(t''+\theta_n)\boldsymbol{z}|} Q_{\rm w}(\boldsymbol{z}) \\ & \times ({\rm i}\omega')^2 S_{\rm inc}(\omega') \frac{{\rm e}^{-{\rm i}\omega'(t''-|\boldsymbol{x}-\mathcal{O}(t''+\theta_n)\boldsymbol{z}|/c)}}{8\pi^2|\boldsymbol{x}-\mathcal{O}(t''+\theta_n)\boldsymbol{z}|}\, {\rm d}\omega' {\rm d}\omega {\rm d}t'' {\rm d}\boldsymbol{z} \,. \end{aligned} \tag{9}$$

We use the far-field approximation $|\boldsymbol{x} - \boldsymbol{w}| = |\boldsymbol{x}| - \hat{\boldsymbol{x}} \cdot \boldsymbol{w} + {\rm O}(|\boldsymbol{x}|^{-1})$ (with the hat denoting unit vector) and the notation $R = |\boldsymbol{x}|$, $\hat{\boldsymbol{R}}(t) = -\mathcal{O}^{\rm T}(t)\hat{\boldsymbol{x}}$ to rewrite (9) as

$$\begin{aligned} s_{\rm sc}(\boldsymbol{x}, n, t) \approx \frac{1}{(8\pi^2 R)^2} \int & Q_{\rm w}(\boldsymbol{z})({\rm i}\omega')^2 S_{\rm inc}(\omega')\, {\rm e}^{-{\rm i}\omega[t-t''-\theta_n-(R+\hat{\boldsymbol{R}}(t''+\theta_n)\cdot\boldsymbol{z})/c]} \\ & \times {\rm e}^{-{\rm i}\omega'[t''-(R+\hat{\boldsymbol{R}}(t''+\theta_n)\cdot\boldsymbol{z})/c]}\, {\rm d}\omega {\rm d}\omega' {\rm d}t'' {\rm d}\boldsymbol{z} \,. \end{aligned} \tag{10}$$

Next we correlate the scattered signal with a signal of the form $s_{\rm inc}(\alpha(t' - t)) = \int S(\omega'') \exp(-{\rm i}\omega''\alpha(t' - t))\, {\rm d}\omega''$ to obtain the output of the correlation receiver:

$$\begin{aligned} \eta_{\rm w}(\theta_n, t, \alpha) &= \int s_{\rm sc}(\boldsymbol{x}, n, t')\overline{s}_{\rm inc}(\alpha(t'-t))\, {\rm d}t' \\ &= \frac{1}{(8\pi^2 R)^2} \int Q_{\rm w}(\boldsymbol{z})({\rm i}\omega')^2 S_{\rm inc}(\omega')\overline{S}_{\rm inc}(\omega'')\, {\rm e}^{-{\rm i}\omega[t'-t''-\theta_n-(R+\hat{\boldsymbol{R}}(t''+\theta_n)\cdot\boldsymbol{z})/c]} \\ &\quad \times {\rm e}^{-{\rm i}\omega'[t''-(R+\hat{\boldsymbol{R}}(t''+\theta_n)\cdot\boldsymbol{z})/c]}\, {\rm e}^{{\rm i}\omega''\alpha(t'-t)}\, {\rm d}\omega {\rm d}\omega' {\rm d}\omega'' {\rm d}t' {\rm d}t'' {\rm d}\boldsymbol{z} \,, \end{aligned} \tag{11}$$

where the bar denotes complex conjugation. In (11) we carry out the integrations over ω and t' to obtain

$$\begin{aligned} \eta_{\rm w}(\theta_n, t, \alpha) = \frac{1}{32\pi^3 R^2} \int & Q_{\rm w}(\boldsymbol{z})\, ({\rm i}\omega')^2 S_{\rm inc}(\omega')\overline{S}_{\rm inc}(\omega'')\, {\rm e}^{-{\rm i}\omega'[t''-(R+\hat{\boldsymbol{R}}(t''+\theta_n)\cdot\boldsymbol{z})/c]} \\ & \times\, {\rm e}^{-{\rm i}\omega''\alpha(t-\theta_n-t''-(R+\hat{\boldsymbol{R}}(t''+\theta_n)\cdot\boldsymbol{z})/c]}\, {\rm d}\omega' {\rm d}\omega'' {\rm d}t'' {\rm d}\boldsymbol{z} \,. \end{aligned} \tag{12}$$

For the remainder of this section we assume that $\ddot{\mathcal{O}}(\theta_n)$ is sufficiently small that we can expand $\mathcal{O}(t''+\theta_n)$ as a function linear in t''. We use this approximation in section 3 to explore the "usual" imaging radar situation corresponding to a signal $s_{\rm inc}(t)$ made up of a series of short pulses. In section 4 we also briefly consider the limiting case of a monochromatic signal made up of a single long-duration pulse.

When rotational acceleration is negligible ($\ddot{\mathcal{O}}(\theta_n) = 0$) we can expand $\mathcal{O}(t''+\theta_n)$ in a Taylor series about $t''=0$, so that $\mathcal{O}(t''+\theta_n) = \mathcal{O}(\theta_n) + \dot{\mathcal{O}}(\theta_n)t''$. We introduce the notation

$$\begin{aligned} &\hat{\boldsymbol{R}}_n \equiv \hat{\boldsymbol{R}}(\theta_n) = -\mathcal{O}^{\rm T}(\theta_n)\hat{\boldsymbol{x}}, \quad r_n(\boldsymbol{z}) \equiv R + \hat{\boldsymbol{R}}_n\cdot\boldsymbol{z}, \quad \text{and} \\ &\nu_n(\boldsymbol{z}) \equiv \hat{\boldsymbol{R}}_n\cdot\mathcal{O}^{\rm T}(\theta_n)\dot{\mathcal{O}}(\theta_n)\boldsymbol{z} = -\dot{\mathcal{O}}^{\rm T}(\theta_n)\hat{\boldsymbol{x}}\cdot\boldsymbol{z}. \end{aligned} \tag{13}$$

The quantity ν is the down-range component of velocity at the point $\boldsymbol{z}$ due to rotation. With the notation (13) we can use the orthogonality of $\mathcal{O}$ to write the term involving $\hat{\boldsymbol{R}}(t''+\theta_n)$ in the exponent of (12) as $\hat{\boldsymbol{R}}(t''+\theta_n)\cdot\boldsymbol{z} = \hat{\boldsymbol{R}}_n\cdot\boldsymbol{z} + t''\nu_n(\boldsymbol{z})$.

Using this expansion for $\hat{\boldsymbol{R}}(t''+\theta_n)\cdot\boldsymbol{z}$ in (12), we find

$$\begin{aligned} \eta_{\rm w}(\theta_n,t,\alpha) = \frac{1}{32\pi^3R^2}\int Q_{\rm w}(\boldsymbol{z})\,({\rm i}\omega')^2 S_{\rm inc}(\omega')\overline{S}_{\rm inc}(\omega'')\,{\rm e}^{-{\rm i}\omega''\alpha[t-\theta_n-t''-(r_n+\nu_n t'')/c]} \\ \times\,{\rm e}^{-{\rm i}\omega'[t''-(r_n+\nu_n t'')/c]}\,{\rm d}\omega'{\rm d}\omega''{\rm d}t''{\rm d}\boldsymbol{z}\,. \end{aligned} \tag{14}$$

Performing the t'' and ω'' integrations in (14) yields

$$\begin{aligned} \eta_{\rm w}(\theta_n,t,\alpha) = \frac{1}{(4\pi R)^2\alpha}\int Q_{\rm w}(\boldsymbol{z})\,\frac{({\rm i}\omega')^2 S_{\rm inc}(\omega')\overline{S}_{\rm inc}\left(\alpha_n'(\boldsymbol{z})\alpha^{-1}\omega'\right)}{1+\nu_n(\boldsymbol{z})/c} \\ \times\,{\rm e}^{-{\rm i}\omega'[\alpha_n'(\boldsymbol{z})(t-\theta_n)-(1+\alpha_n'(\boldsymbol{z}))r_n(\boldsymbol{z})/c]}\,{\rm d}\omega'{\rm d}\boldsymbol{z}\,, \end{aligned} \tag{15}$$

where

$$\alpha_n'(\boldsymbol{z}) \equiv \frac{1-\nu_n(\boldsymbol{z})/c}{1+\nu_n(\boldsymbol{z})/c} = 1 - \frac{2\nu_n(\boldsymbol{z})}{c+\nu_n(\boldsymbol{z})}\,. \tag{16}$$

We interpret α_n' as the *Doppler scale factor*. Equation (15) is our model for the radar data in the weak-scattering case. We note that (15) expresses the data $\eta_{\rm w}$ as a Fourier Integral Operator acting on $Q_{\rm w}$.

We remark also that the ω' integral of (15),

$$\mathcal{A}\left(\alpha_n't-t',\alpha_n'\alpha^{-1}\right) = \int({\rm i}\omega')^2 S_{\rm inc}(\omega')\overline{S}_{\rm inc}\left(\alpha_n'\alpha^{-1}\omega'\right){\rm e}^{-{\rm i}\omega'(\alpha_n't-t')}\,{\rm d}\omega'\,, \tag{17}$$

is an imaging kernel related to the radar ambiguity function and, in the presence of measurement noise, determines the resolution to which $Q_{\rm w}(\boldsymbol{z})$ can be estimated [8].

2.2. Multiple scattering

Multiple scattering does not fit into the weak-scattering model. In the case where there are only two isotropic point scatterers, we use the exact solution derived in Appendix A for the scattered field due to the incident wave (6). We consider the case of a rotating target; i.e., we replace $\boldsymbol{z}^j$ of (A.8) by $\mathcal{O}(t')\boldsymbol{z}^j$:

$$\begin{aligned} u^{\rm sc}(t,\boldsymbol{x}) = \sum_{j=1}^{2}\frac{1}{8\pi^2}\int g(t-t',|\boldsymbol{x}-\mathcal{O}(t')\boldsymbol{z}^j|)\frac{\mu_j S_{\rm inc}(\omega')}{1-\mu_1\mu_2\,{\rm e}^{{\rm i}2\omega'L}/(4\pi L)^2} \\ \times\left(\frac{{\rm e}^{{\rm i}\omega'|\boldsymbol{x}-\mathcal{O}(t')\boldsymbol{z}^j|/c}}{|\boldsymbol{x}-\mathcal{O}(t')\boldsymbol{z}^j|} + \mu_{j'}\frac{{\rm e}^{2{\rm i}\omega'L}}{4\pi L}\frac{{\rm e}^{{\rm i}\omega'|\boldsymbol{x}-\mathcal{O}(t')\boldsymbol{z}^{j'}|/c}}{|\boldsymbol{x}-\mathcal{O}(t')\boldsymbol{z}^{j'}|}\right){\rm e}^{-{\rm i}\omega'(t'-\theta_n)}{\rm d}\omega'{\rm d}t'\,, \end{aligned} \tag{18}$$

where $j' = 1$ if $j = 2$ and $j' = 2$ if $j = 1$. Equation (18) is simplified as in section 2: we use the oscillatory-integral representation (4) for g; make the far-field approximation; apply the change of variables $t'' = t' - \theta_n$; expand $\mathcal{O}$ in a Taylor series; and use the notation defined above (10). With these substitutions we obtain

$$u^{\mathrm{sc}}(\boldsymbol{x}, n, t) = \frac{1}{(8\pi^2 R)^2} \sum_{\substack{j=1,2 \\ j' \neq j}} \int \frac{\mu_j S_{\mathrm{inc}}(\omega')\, \mathrm{e}^{-\mathrm{i}\omega[t - t'' - \theta_n - (r_n(\boldsymbol{z}^j) + \nu_n(\boldsymbol{z}^j)t'')/c]}}{1 - \mu_j \mu_{j'}\, \mathrm{e}^{\mathrm{i}2\omega' L}/(4\pi L)^2} \tag{19}$$

$$\times \left[\mathrm{e}^{\mathrm{i}\omega'[t'' - (r_n(\boldsymbol{z}^j) + \nu_n(\boldsymbol{z}^j)t'')/c]} + \mu_{j'} \frac{\mathrm{e}^{\mathrm{i}\omega' L}}{4\pi L}\, \mathrm{e}^{\mathrm{i}\omega'[t'' - (r_n(\boldsymbol{z}^{j'}) + \nu_n(\boldsymbol{z}^{j'})t'')/c]} \right] \mathrm{d}\omega' \mathrm{d}\omega\, \mathrm{d}t'' \,,$$

where $L \equiv |\boldsymbol{z}^j - \boldsymbol{z}^{j'}|$.

The output of the correlation receiver is

$$\eta_{\mathrm{mult}}(\theta_n, t, \alpha) = \frac{1}{(4\pi R)^2 \alpha} \sum_{\substack{j=1,2 \\ j' \neq j}} \int \frac{\mu_j S_{\mathrm{inc}}(\omega')}{(1 - \mu_j \mu_{j'}\, \mathrm{e}^{\mathrm{i}2\omega' L}/(4\pi L)^2)\,(1 + \nu_n(\boldsymbol{z}^j)/c)} \tag{20}$$

$$\times \left[\overline{S}_{\mathrm{inc}}\left(\alpha^j_{j;n} \alpha^{-1} \omega'\right) \mathrm{e}^{-\mathrm{i}\omega'[\alpha^j_{j;n} t_n - (1 + \alpha^j_{j;n}) r_n(\boldsymbol{z}^j)/c]} \right.$$

$$\left. + \frac{\mu_{j'}}{4\pi L} \overline{S}_{\mathrm{inc}}\left(\alpha^{j'}_{j;n} \alpha^{-1} \omega'\right) \mathrm{e}^{-\mathrm{i}\omega'[\alpha^{j'}_{j;n} t_n - ((1 + \alpha^{j'}_{j;n}) r_n(\boldsymbol{z}^j) + L + \hat{\boldsymbol{R}}_n \cdot (\boldsymbol{z}^{j'} - \boldsymbol{z}^j))/c]} \right] \mathrm{d}\omega' \,,$$

where

$$\alpha^l_{j;n} \equiv \frac{1 - \nu_n(\boldsymbol{z}^l)/c}{1 + \nu_n(\boldsymbol{z}^j)/c} \,. \tag{21}$$

(Observe that when $\mu_j \mu_{j'} \ll \mu_j \leq 1$, (20) reduces to (15) for $Q_{\mathrm{mult}}(\boldsymbol{z}) = \mu_1 \delta(\boldsymbol{z} - \boldsymbol{z}^1) + \mu_2 \delta(\boldsymbol{z} - \boldsymbol{z}^2)$.)

Expanding the denominator of (20), retaining only terms cubic and lower in μ_j, and simplifying, we obtain

$$\eta_{\mathrm{mult}}(\theta_n, t, \alpha) \approx \frac{1}{(4\pi R)^2 \alpha} \sum_{\substack{j=1,2 \\ j' \neq j}} \int \frac{\mu_j}{1 + \nu_n(\boldsymbol{z}^j)/c} S_{\mathrm{inc}}(\omega') \tag{22}$$

$$\times \left[\overline{S}_{\mathrm{inc}}\left(\alpha^j_{j;n} \alpha^{-1} \omega'\right) \mathrm{e}^{-\mathrm{i}\omega'[\alpha^j_{j;n} t_n - (1 + \alpha^j_{j;n}) r_n(\boldsymbol{z}^j)/c]} \right.$$

$$+ \frac{\mu_{j'}}{4\pi L} \overline{S}_{\mathrm{inc}}\left(\alpha^{j'}_{j;n} \alpha^{-1} \omega'\right) \mathrm{e}^{-\mathrm{i}\omega'[\alpha^{j'}_{j;n} t_n - (r_n(\boldsymbol{z}^{j'}) + \alpha^{j'}_{j;n} r_n(\boldsymbol{z}^j) + L)/c]}$$

$$\left. + \frac{\mu_j \mu_{j'}}{(4\pi L)^2} \overline{S}_{\mathrm{inc}}\left(\alpha^j_{j;n} \alpha^{-1} \omega'\right) \mathrm{e}^{-\mathrm{i}\omega'[\alpha^j_{j;n} t_n - ((1 + \alpha^j_{j;n}) r_n(\boldsymbol{z}^j) + 2L)/c]} \right] \mathrm{d}\omega' \,.$$

Equation (22) is our model for radar data in the multiple-scattering case. We note that (22) is a sum of oscillatory integrals, to which the techniques of microlocal analysis can be applied.

Our multiple-scattering model (22) differs significantly from that of the weak-scattering case in that additional bookkeeping must be performed to account for target substructure position relative to other scatterers. In addition, the multiple-scattering expression depends on the overall target orientation and involves multiplicative terms of the form $\exp(\mathrm{i}\omega m L/c)$ (for some integer m).

2.3. Dispersive scattering by reentrant structures

For reentrant structures with openings that can be associated with the location $\boldsymbol{z}$, the most complicated aspect of multiple scattering (i.e., the accounting) can be eliminated. This simplification is made possible by a model [3] for scattering from such structures that includes wave propagation within the duct or cavity. Here, the analysis is done by treating the reentrant structure as a waveguide: for Q in (9), we use

$$Q_{\mathrm{d}}(\omega, \theta_n, \boldsymbol{z}) = q_{\mathrm{M}}(\theta_n, \boldsymbol{z}) \sum_m \rho_m \, \mathrm{e}^{\mathrm{i}2L(\boldsymbol{z})c^{-1}\sqrt{\omega^2 - w_m^2}} \,. \tag{23}$$

In this equation, m indexes the eigen-solutions (modes) of the waveguide problem, w_m denotes the mode cutoff frequency, ρ_m is the strength of the mode, q_{M} is proportional to the amount of energy that gets coupled into the reentrant feature, and $L(\boldsymbol{z})$ is the distance from the mouth of the duct/cavity to a scattering center within. We denote by M the mouth of the structure and assume that $L(\boldsymbol{z})$ is constant over M and zero off M. We note that this scattering model includes dependencies on ω and θ.

We take

$$q_{\mathrm{M}}(\theta_n, \boldsymbol{z}) = A(\hat{\boldsymbol{N}} \cdot \hat{\boldsymbol{R}}_n)\varphi_{\mathrm{M}}(\boldsymbol{z}) \,, \tag{24}$$

where $\hat{\boldsymbol{N}}$ is the (effective) normal to the waveguide opening, A is a coupling pattern that gives the angular dependence of the coupling strength, and φ_{M} is a function that is supported in a neighborhood of M and is further characterized in section 3.4. Equation (23) models only the contribution to the scattered field from scatterers within the waveguide; scattering from the edges of the waveguide mouth is handled separately (as in sections 2.1 or 2.2).

In the time domain, (23) corresponds to

$$\begin{aligned} q_{\mathrm{d}}(t', \theta_n, \boldsymbol{z}) &= q_{\mathrm{M}}(\theta_n, \boldsymbol{z}) \sum_m \rho_m \int \mathrm{e}^{\mathrm{i}2L(\boldsymbol{z})c^{-1}\sqrt{\omega^2 - w_m^2}} \, \mathrm{e}^{-\mathrm{i}\omega t'} \, \mathrm{d}\omega \\ &= q_{\mathrm{M}}(\theta_n, \boldsymbol{z}) \sum_m \rho_m I_m(t') \,. \end{aligned} \tag{25}$$

Since $L(\boldsymbol{z})$ is assumed to be constant on M,

$$I_m(t') = \int \mathrm{e}^{\mathrm{i}2Lc^{-1}\sqrt{\omega^2 - w_m^2}} \, \mathrm{e}^{-\mathrm{i}\omega t'} \, \mathrm{d}\omega \tag{26}$$

is independent of $\boldsymbol{z}$. This integral can be expressed in terms of the Heaviside function H and the Bessel function J_0 as

$$\begin{aligned} I_m(t') = \int & H(t'' - 2L/c) J_0 \left(w_m \sqrt{t''^2 - (2L/c)^2} \right) \\ & \times \int \mathrm{i}\sqrt{\omega^2 - w_m^2} \, \mathrm{e}^{-\mathrm{i}\omega(t' - t'')} \, \mathrm{d}\omega \mathrm{d}t'' \,. \end{aligned} \tag{27}$$

Consequently, $I_m(t)$ is the convolution of $H(t - 2L/c) J_0 \left(w_m \sqrt{t^2 - (2L/c)^2} \right)$ with

$$f(t) = \mathrm{i}\mathcal{F}^{-1} \left\{ \sqrt{\omega^2 - w_m^2} \right\} (t) \,. \tag{28}$$

Since the downrange dimension of a typical radar image is actually travel time, we can see from (27) that the image of scattering centers located within ducts/cavities that obey this model will not be localized to a point. Instead, the associated image will be stretched and extended in the downrange dimension. The "stretching" property follows from the scaling behavior of w_m in the argument of J_0. This general behavior is a consequence of *dispersion*—waves reflected from such scattering centers exhibit a frequency-dependent time delay (as in

equation (25)). In practice, such nonlocal image elements can be difficult to map to the local target structures that created them, and are usually considered to be image artifacts.

To obtain a model for radar data, we substitute for Q in (9) the expression

$$Q_{\mathrm{d}}(\omega', \theta_n, \boldsymbol{z}) = \int q_{\mathrm{d}}(t', \theta_n, \boldsymbol{z})\, \mathrm{e}^{\mathrm{i}\omega' t'}\, \mathrm{d}t'. \tag{29}$$

We carry out the computations (10) through (15) as before and, finally, in (15) we substitute expression (29). We thus obtain for the output of the correlation receiver

$$\eta_{\mathrm{d}}(\theta_n, t, \alpha) = \int K_{\mathrm{d}}(\theta_n, t, \alpha, \boldsymbol{z}, t') q_{\mathrm{d}}(t', \theta_n, \boldsymbol{z})\, \mathrm{d}t' \mathrm{d}\boldsymbol{z}\,, \tag{30}$$

where

$$K_{\mathrm{d}}(\theta_n, t, \alpha, \boldsymbol{z}, t') = \frac{1}{(4\pi R)^2 \alpha} \int \frac{(\mathrm{i}\omega')^2 S_{\mathrm{inc}}(\omega') \overline{S}_{\mathrm{inc}}(\alpha_n' \alpha^{-1} \omega')}{1 + \nu_n / c} \times \mathrm{e}^{-\mathrm{i}\omega'[\alpha_n' t_n - (1+\alpha_n') r_n / c - t']}\, \mathrm{d}\omega'\,. \tag{31}$$

This equation is our model for radar data from structurally dispersive target elements. Again, this result involves oscillatory integrals which can be studied with the techniques of microlocal analysis. Although equation (30) is not quite in the form of a Fourier Integral Operator (because it involves mutliplication in the θ_n variable as well as integration), it could be converted into one by introducing another variable and modifying the phase of (31) appropriately.

3. Wavefront sets for radar data

We focus on localized scattering centers such as corners, specular "flashes" from smooth surfaces, and re-entrant structures such as ducts and engine inlets. These target features we characterize by the singular structure of Q, which we describe in terms of its wavefront set.

3.1. Wavefront sets

Mathematically the singular structure of a function can be characterized by its *wavefront set*, which involves both the location $\boldsymbol{x}$ and corresponding directions $\boldsymbol{\xi}$ of singularities [9, 13, 30, 32].

Definition. The point $(\boldsymbol{x}_0, \boldsymbol{\xi}_0)$ is *not* in the wavefront set $\mathrm{WF}(f)$ of the function f if there is a smooth cutoff function ψ with $\psi(\boldsymbol{x}_0) \neq 0$, for which the Fourier transform $\mathcal{F}(f\psi)(\lambda\boldsymbol{\xi})$ decays rapidly (i.e., faster than any polynomial) as $\lambda \to \infty$ for $\boldsymbol{\xi}$ in a neighborhood of $\boldsymbol{\xi}_0$.

This definition says that to determine whether $(\boldsymbol{x}_0, \boldsymbol{\xi}_0)$ is in the wavefront set of f, one should 1) localize around $\boldsymbol{x}_0$ by multiplying by a smooth function ψ supported in the neighborhood of $\boldsymbol{x}_0$, 2) Fourier transform $f\psi$, and 3) examine the decay of the Fourier transform in the direction $\boldsymbol{\xi}_0$. Rapid decay of the Fourier transform in direction $\boldsymbol{\xi}_0$ corresponds to smoothness of the function f in the direction $\boldsymbol{\xi}_0$ [16].

Example: a point scatterer. If $Q(\boldsymbol{x}) = \delta(\boldsymbol{x})$, then $\mathrm{WF}(Q) = \{(\boldsymbol{0}, \boldsymbol{\xi}) : \boldsymbol{\xi} \neq \boldsymbol{0}\}$.

Example: a specular flash. Suppose $Q(\boldsymbol{x}) = H(\boldsymbol{x} \cdot \boldsymbol{\nu})$, where H denotes the Heaviside function. Then $\mathrm{WF}(Q) = \{(\boldsymbol{x}, \alpha\boldsymbol{\nu}) : \boldsymbol{x} \cdot \boldsymbol{\nu} = 0, \alpha \neq 0\}$.

Wavefront sets can be specified closed sets ([14], p. 255):

Theorem 1 *If $S = \{(\boldsymbol{x}, \boldsymbol{\xi})\}$ is a closed subset of $\mathbb{R}^n \times (\mathbb{R}^n \setminus 0)$, then there is a function on $\mathbb{R}^n$ whose wavefront set is S.*

Our strategy is to work out explicitly how the wavefront set of Q corresponds (via (12)) to the wavefront set of η. We take the wavefront set of Q to be

$$\mathrm{WF}(Q) = \{(\boldsymbol{z}, \boldsymbol{\zeta}) : \boldsymbol{\zeta} \neq \mathbf{0}\} \ . \tag{32}$$

To compute the wavefront set of η, we use the following four theorems [9].

Theorem 2 *(Wavefront set of an oscillatory integral) Suppose K is defined by*

$$K(\boldsymbol{x}) = \int \mathrm{e}^{\mathrm{i}\omega\,\phi(\boldsymbol{x})} a(\boldsymbol{x}, \omega)\, \mathrm{d}\omega, \tag{33}$$

where a satisfies the following condition: there is some μ and M for which, on any compact set X, the estimate

$$|\partial_{x_1}^{n_1} \partial_{x_2}^{n_2} \cdot \partial_{x_N}^{n_N} \partial_\omega^m a(\boldsymbol{x}, \omega)| \leq C_{X,n,m} (1 + |\omega|)^{\mu - Mm + (1-M)|\boldsymbol{n}|} \tag{34}$$

holds, with $|\boldsymbol{n}| = \sum n_j$. Then the wavefront set of K satisfies

$$\mathrm{WF}(K) \subseteq \{(\boldsymbol{x}, \nabla\phi) : \phi(x) = 0\}. \tag{35}$$

Theorem 3 *(Wavefront set of a product) Suppose*

$$\mathrm{WF}(f) + \mathrm{WF}(g) \equiv \{(\boldsymbol{x}, \boldsymbol{\xi}_f + \boldsymbol{\xi}_g) : (\boldsymbol{x}, \boldsymbol{\xi}_f) \in \mathrm{WF}(f),\ (\boldsymbol{x}, \boldsymbol{\xi}_g) \in \mathrm{WF}(g)\} \tag{36}$$

contains no points of the form $(\boldsymbol{x}, \mathbf{0})$. Then the wavefront set of the product fg satisfies

$$\mathrm{WF}(fg) \subseteq (\mathrm{WF}(f) + \mathrm{WF}(g)) \cup \mathrm{WF}(f) \cup \mathrm{WF}(g)\,. \tag{37}$$

Definition. Suppose P is a mapping from X to Y, where X and Y are assumed to be smooth manifolds, and f is a function defined on Y. Then the *pull-back* $P^* f$ is a function on X defined by $P^* f(x) = f(P(x))$.

We can also extend this notion to apply to distributions f defined on Y, provided the wavefront set of f avoids the "bad" set

$$\left\{(\boldsymbol{y}, \boldsymbol{\eta}) : \boldsymbol{y} = P(\boldsymbol{x}), DP_{\boldsymbol{x}}^{\mathrm{T}} \boldsymbol{\eta} = \mathbf{0} \text{ for some } \boldsymbol{x} \in X\right\} \ . \tag{38}$$

Theorem 4 *(Pull-back of a wavefront set) Suppose P is a smooth mapping from X to Y, and f is a distribution on Y whose wavefront set avoids the set (38). Then the wavefront set of $P^* f$ is contained in*

$$\mathrm{WF}(P^* f) \subseteq \left\{(\boldsymbol{x}, \boldsymbol{\xi}) : \boldsymbol{\xi} = DP_{\boldsymbol{x}}^{\mathrm{T}} \boldsymbol{\eta} \text{ for some } \boldsymbol{\eta} \text{ such that } (P(\boldsymbol{x}), \boldsymbol{\eta}) \in \mathrm{WF}(f)\right\}. \tag{39}$$

Application to embedding a function in a larger space. If we have a function f of the variable x, and we want to consider it to be a function of the variables x and y, then we can write f as the pull-back $P^* f$ for the mapping $P : (x, y) \mapsto x$. Then, since the Jacobian of P is $DP = (1, 0)$, the wavefront set of $P^* f$ is

$$\begin{aligned} \mathrm{WF}(P^* f) &= \left\{((x, y); DP^{\mathrm{T}} \xi) : (x, \xi) \in \mathrm{WF}(f)\right\} \\ &= \{(x, y); (\xi, 0)) : (x, \xi) \in \mathrm{WF}(f)\} \ . \end{aligned} \tag{40}$$

Definition. If g is a distribution defined on X, the *push-forward* P_*g of g satisfies $P_*g = (P^*)^{\mathrm{T}}g$, where the superscript T denotes transpose; in other words, for any test function f, $\langle P_*g, f\rangle = \langle g, P^*f\rangle$.

Theorem 5 *(Push-forward of a wavefront set) Suppose P is a smooth mapping from X to Y, and g is a distribution defined on X. Then*

$$\mathrm{WF}(P_*g) \subseteq \{(\boldsymbol{y}, \boldsymbol{\eta}) : \boldsymbol{y} = P(\boldsymbol{x}) \,\mathrm{with}\, (\boldsymbol{x}, DP^{\mathrm{T}}\boldsymbol{\eta}) \in \mathrm{WF}(g)\}\,. \tag{41}$$

Application to calculating wavefront sets of integrals. We use push-forwards in the following way. Suppose we have a distribution g in the variables x and y. Then we can interpret $\int g(x,y)\,\mathrm{d}y$ as the push-forward P_*g for the mapping $P : (x,y) \mapsto x$, because $\int g(x,y)f(x)\,\mathrm{d}x\mathrm{d}y = \langle g, P^*f\rangle_{x,y} = \langle P_*g, f\rangle_x = \int (P_*g)(x)f(x)\,\mathrm{d}x$.

3.2. Wavefront set for the (traditional) weak scattering case

In (15), we let $t_n \equiv t - \theta_n$ denote the *fast time* (similarly, θ_n is the *slow time*). Then we can write (15) as

$$\eta_{\mathrm{w}}(\theta_n, t, \alpha) = \int K_{\mathrm{w}}(\theta_n, t, \alpha, \boldsymbol{z}) Q_{\mathrm{w}}(\boldsymbol{z})\,\mathrm{d}\boldsymbol{z}\,, \tag{42}$$

where

$$K_{\mathrm{w}}(\theta_n, t, \alpha, \boldsymbol{z}) = \frac{1}{(4\pi R)^2\alpha}\int \frac{(\mathrm{i}\omega')^2 S_{\mathrm{inc}}(\omega')\overline{S}_{\mathrm{inc}}(\alpha_n'\alpha^{-1}\omega')}{1+\nu_n/c} \times \mathrm{e}^{-\mathrm{i}\omega'[\alpha_n' t_n - (1+\alpha_n')r_n/c]}\,\mathrm{d}\omega'\,. \tag{43}$$

Under the assumptions on S_{inc} of Theorem 2, equation (42) expresses η_{w} in terms of a Fourier Integral Operator applied to Q_{w}, and therefore the wavefront set of η_{w} can be calculated in terms of that of Q_{w} by standard means [9, 13, 32]. We illustrate here an alternative method for calculating the wavefront set that we will use in section 2.3.

This alternative approach considers η_{w} to be a push-forward of the product $K_{\mathrm{w}}Q_{\mathrm{w}}$, and we use the theorems of section 3.1. More specifically, η_{w} is formed from two operations: first we multiply Q_{w} by K_{w}, and then push forward the product by means of the projection operator $\mathcal{P} : (\theta_n, t, \alpha, \boldsymbol{z}) \mapsto (\theta_n, t, \alpha)$. In the process of multiplying Q_{w} by K_{w}, we need to consider Q_{w} to be a function of the same variables as K_{w}; for this we consider the pull-back of Q_{w}. Calculation of the wavefront set of η_{w} can therefore be carried out in the following steps: a) calculate the wavefront set of K_{w} from theorem 2; b) calculate the wavefront set of $K_{\mathrm{w}}Q_{\mathrm{w}}$ from theorems 4 and 3; and c) calculate the wavefront set of $\eta_{\mathrm{w}} = P_*(K_{\mathrm{w}}Q_{\mathrm{w}})$ from theorem 5.

Step a) We assume that $(\mathrm{i}\omega')^2 S_{\mathrm{inc}}(\omega')\overline{S}_{\mathrm{inc}}(\alpha_n'\alpha^{-1}\omega')$ satisfies the hypothesis of theorem 2. The phase of K_{w} is

$$\phi = -\omega'[\alpha_n'(\boldsymbol{z})t_n - (1+\alpha_n'(\boldsymbol{z}))r_n(\boldsymbol{z})/c] \tag{44}$$

and so

$$\begin{aligned}\mathrm{WF}(K_{\mathrm{w}}) \subseteq \Big\{ &(\theta_n, t, \alpha, \boldsymbol{z}; \sigma, \tau, \gamma, \boldsymbol{\zeta}) : \\ &\alpha_n'(\boldsymbol{z})t_n - (1+\alpha_n'(\boldsymbol{z}))r_n(\boldsymbol{z})/c = 0, \\ &\sigma = \partial_{\theta_n}\phi = \omega'\left[\alpha_n'(\boldsymbol{z}) - (1+\alpha_n'(\boldsymbol{z}))\dot{\mathcal{O}}^{\mathrm{T}}(\theta_n)\hat{\boldsymbol{x}}\cdot\boldsymbol{z}/c\right],\end{aligned} \tag{45}$$

$$\tau = \partial_t \phi = -\omega' \alpha_n'(\boldsymbol{z}),$$
$$\gamma = \partial_\alpha \phi = 0,$$
$$\boldsymbol{\zeta} = \nabla_z \phi = -\omega' \left[\frac{2\dot{\mathcal{O}}^{\mathrm{T}}(\theta_n)\hat{\boldsymbol{x}}}{c(1+\nu_n(\boldsymbol{z})/c)^2}(t_n - r_n(\boldsymbol{z})/c) + \frac{1}{c}(1+\alpha_n'(\boldsymbol{z}))\mathcal{O}^{\mathrm{T}}(\theta_n)\hat{\boldsymbol{x}} \right] \Bigg\},$$

where (as before) we have taken $\ddot{\mathcal{O}}(\theta_n) \approx 0$. Some of the details of the calculations can be found in Appendix B. (Note that for these short-duration signals, ϕ will be independent of α and so $\gamma = 0$ for all cases considered in the remainder of section 3.)

We can simplify the expressions for σ and $\boldsymbol{\zeta}$ in (45) as follows. First, a straightforward computation shows that

$$\sigma = \omega' \left(\alpha_n'(\boldsymbol{z}) + \frac{2\nu_n(\boldsymbol{z})}{c+\nu_n(\boldsymbol{z})} \right) = \omega' \,. \tag{46}$$

Then, to simplify $\boldsymbol{\zeta}$, we use the fact that from the criticality condition we have $r_n c^{-1} = \alpha_n' t_n (1+\alpha_n')^{-1}$. Substitution of this relation into the result for $\boldsymbol{\zeta}$ together with the facts $1 - [\alpha_n'/(1+\alpha_n')] = (1+\nu/c)/2$ and $1+\alpha_n' = 2/(1+\nu/c)$ yields

$$\begin{aligned} \boldsymbol{\zeta} &= \frac{-2\omega'}{c+\nu_n(\boldsymbol{z})} \left[\mathcal{O}^{\mathrm{T}}(\theta_n)\hat{\boldsymbol{x}} + \dot{\mathcal{O}}^{\mathrm{T}}(\theta_n)\hat{\boldsymbol{x}}\tfrac{1}{2}t_n \right] \\ &= \frac{2\omega'}{c+\nu_n(\boldsymbol{z})} \left[\hat{\boldsymbol{R}}_n + \frac{\mathrm{d}\hat{\boldsymbol{R}}_n}{\mathrm{d}\theta_n}\tfrac{1}{2}t_n \right] \,. \end{aligned} \tag{47}$$

The Taylor series expansion for $\hat{\boldsymbol{R}}(t)$ obeys $\hat{\boldsymbol{R}}(\frac{1}{2}t_n) \approx \hat{\boldsymbol{R}}_n + \frac{1}{2}t_n\, \mathrm{d}\hat{\boldsymbol{R}}_n/\mathrm{d}\theta_n$ and so we can write

$$\boldsymbol{\zeta} = \frac{2\omega'}{c+\nu_n(\boldsymbol{z})}\hat{\boldsymbol{R}}(\tfrac{1}{2}t_n)\,. \tag{48}$$

Step b) By theorem 3, the wavefront set of the product of K_{w} and Q_{w} obeys

$$\mathrm{WF}(K_{\mathrm{w}}Q_{\mathrm{w}}) \subseteq (\mathrm{WF}(K_{\mathrm{w}}) + \mathrm{WF}(Q_{\mathrm{w}})) \cup \mathrm{WF}(K_{\mathrm{w}}) \cup \mathrm{WF}(Q_{\mathrm{w}})\,. \tag{49}$$

We write the wavefront set of Q_{w} in $\mathbb{R}^6$ as $\mathrm{WF}^6(Q_{\mathrm{w}}) = \left\{ (\tilde{\boldsymbol{z}};\tilde{\boldsymbol{\zeta}}) : \tilde{\boldsymbol{\zeta}} \neq \boldsymbol{0} \right\}$; pulled back (via Theorem 4) to $\mathbb{R}^{12}$ it is

$$\mathrm{WF}^{12}(Q_{\mathrm{w}}) = \left\{ (\theta_n, t, \alpha, \tilde{\boldsymbol{z}}; 0,0,0,\tilde{\boldsymbol{\zeta}}) : \theta_n, t, \alpha \in \mathbb{R}, (\boldsymbol{z},\tilde{\boldsymbol{\zeta}}) \in \mathrm{WF}^6(Q_{\mathrm{w}}) \right\}\,. \tag{50}$$

Consequently, we can write

$$\begin{aligned} \mathrm{WF}(K_{\mathrm{w}}) + \mathrm{WF}^{12}(Q_{\mathrm{w}}) \subseteq \Bigg\{ &(\theta_n, t, \alpha, \boldsymbol{z}; \check{\sigma}, \check{\tau}, \check{\gamma} = 0, \check{\boldsymbol{\zeta}}) : \\ &\alpha_n'(\tilde{\boldsymbol{z}})t_n - (1+\alpha_n'(\tilde{\boldsymbol{z}}))r_n(\tilde{\boldsymbol{z}})/c = 0 \quad \text{and} \quad \tilde{\boldsymbol{z}} = \boldsymbol{z}, \\ &\check{\sigma} = \omega',\ \check{\tau} = -\omega'\alpha_n'(\tilde{\boldsymbol{z}}), \\ &\check{\boldsymbol{\zeta}} = \boldsymbol{\zeta} + \tilde{\boldsymbol{\zeta}} = \frac{2\omega'}{c+\nu_n(\tilde{\boldsymbol{z}})}\hat{\boldsymbol{R}}(\tfrac{1}{2}t_n) + \tilde{\boldsymbol{\zeta}} \quad \text{for some } (\tilde{\boldsymbol{z}},\tilde{\boldsymbol{\zeta}}) \in \mathrm{WF}(Q_{\mathrm{w}}) \Bigg\}, \end{aligned} \tag{51}$$

where $\tilde{\boldsymbol{z}}$ and $\tilde{\boldsymbol{\zeta}}$ are defined by (50). Note that $(\check{\sigma}, \check{\tau}, \check{\gamma}, \check{\boldsymbol{\zeta}}) \neq \boldsymbol{0}$ because $\omega' = 0$ is excluded (since $S_{\mathrm{inc}}(\omega')$ has no DC component).

Step c) The Jacobian matrix of the projection $\mathcal{P} : (\theta_n, t, \alpha, \boldsymbol{z}) \mapsto (\theta_n, t, \alpha)$ is

$$D\mathcal{P} = (\mathbf{I}_{3\times 3}\mathbf{0}_{3\times 3}) \tag{52}$$

(assuming $\boldsymbol{z} \in \mathbb{R}^3$).

From theorem 5, the wavefront set of $\eta_{\rm w} = \mathcal{P}_*(K_{\rm w}Q_{\rm w})$ obeys $\mathrm{WF}(\mathcal{P}_*(K_{\rm w}Q_{\rm w})) \subseteq \big\{(\theta_n, t, \alpha; \sigma'', \tau'', \gamma'') : (\theta_n, t, \alpha, \boldsymbol{z}; (D\mathcal{P})^{\rm T}(\sigma'', \tau'', \gamma'')) \in \mathrm{WF}(K_{\rm w}Q_{\rm w})\big\}$. In order to determine when $(D\mathcal{P})^{\rm T}(\sigma'', \tau'', \gamma'') \in \mathrm{WF}(K_{\rm w}Q_{\rm w})$, we note that

$$(D\mathcal{P})^{\rm T}\begin{pmatrix}\sigma''\\ \tau''\\ \gamma''\end{pmatrix} = \begin{pmatrix}\mathbf{I}_{3\times 3}\\ \mathbf{0}_{3\times 3}\end{pmatrix}\begin{pmatrix}\sigma''\\ \tau''\\ \gamma''\end{pmatrix} = \begin{pmatrix}\sigma''\\ \tau''\\ \gamma''\\ \mathbf{0}\end{pmatrix}. \tag{53}$$

In other words, contributions to $\mathrm{WF}(\eta_{\rm w})$ come from elements of (49) with $\check{\boldsymbol{\zeta}} = \mathbf{0}$. There are no such elements in $\mathrm{WF}(K_{\rm w})$ because $\hat{\boldsymbol{R}}(\frac{1}{2}t_n) \neq \mathbf{0}$ in (48). Similarly, there is no contribution from $\mathrm{WF}(Q_{\rm w})$ because elements with $\tilde{\boldsymbol{\zeta}} = \mathbf{0}$ do not appear in (50). Comparison of (53) with (51), on the other hand, yields

$$\tilde{\boldsymbol{\zeta}} = \frac{-2\omega'}{c + \nu_n(\tilde{\boldsymbol{z}})}\hat{\boldsymbol{R}}(\tfrac{1}{2}t_n)\,. \tag{54}$$

This relation describes specular reflection.

Summary For a weakly scattering target, the wavefront set of $\eta_{\rm w}$ is contained in the set ‡

$$\Big\{(\theta_n, t, \alpha; \sigma'', \tau'', \gamma'' = 0) :\ \alpha_n'(\tilde{\boldsymbol{z}})t_n - (1 + \alpha_n'(\tilde{\boldsymbol{z}}))r_n(\tilde{\boldsymbol{z}})/c = 0,\ \sigma'' = \omega', \tag{55}$$

$$\tau'' = -\omega'\alpha_n'(\tilde{\boldsymbol{z}}),\ \text{for which } \tilde{\boldsymbol{\zeta}} = \frac{-2\omega'}{c + \nu_n(\tilde{\boldsymbol{z}})}\hat{\boldsymbol{R}}(\tfrac{1}{2}t_n) \text{ for some } (\tilde{\boldsymbol{z}}, \tilde{\boldsymbol{\zeta}}) \in \mathrm{WF}(Q_{\rm w})\Big\}.$$

In particular, the wavefront set corresponding to a single point scatterer at $\boldsymbol{z}^0$ will be the curve $\alpha_n'(\boldsymbol{z}^0)t_n - (1 + \alpha_n'(\boldsymbol{z}^0))r_n(\boldsymbol{z}^0)/c = 0$ whose normal vector is $(\sigma'', \tau'') \propto \big(1, -\alpha_n'(\boldsymbol{z}^0)\big)$.

3.3. Wavefront sets for multiple scattering

In the case of the two isotropic point scatters that we modeled in section 2.2, the target is simply a sum of two delta functions $Q_{\rm mult}(\boldsymbol{z}) = \delta(\boldsymbol{z} - \boldsymbol{z}^1) + \delta(\boldsymbol{z} - \boldsymbol{z}^2)$. The corresponding wavefront set is

$$\mathrm{WF}(Q_{\rm mult}) = \big\{(\boldsymbol{z}^1, \boldsymbol{\zeta}) :\ \text{all } \boldsymbol{\zeta} \neq \mathbf{0}\big\} \cup \big\{(\boldsymbol{z}^2, \boldsymbol{\zeta}) :\ \text{all } \boldsymbol{\zeta} \neq \mathbf{0}\big\}\ . \tag{56}$$

We see from (22) that multiple-scattering data can be expressed as a sum of oscillatory integrals $\eta_{\rm mult} \approx \eta_1 + \eta_2 + \eta_3$; to each we can simply apply Theorem 2. The corresponding phases are

$$\begin{aligned}\phi_1 &= -\omega'(\alpha_{j;n}^j t_n - (1 + \alpha_{j;n}^j)r_n(\boldsymbol{z}^j)/c) \\ \phi_2 &= -\omega'(\alpha_{j;n}^{j'} t_n - (r_n(\boldsymbol{z}^{j'}) + \alpha_{j;n}^{j'} r_n(\boldsymbol{z}^j) + L)/c) \\ \phi_3 &= -\omega'(\alpha_{j;n}^j t_n - ((1 + \alpha_{j;n}^j)r_n(\boldsymbol{z}^j) + 2L)/c)\,.\end{aligned} \tag{57}$$

‡ We have constructed explicitly the action on $\mathrm{WF}(Q_{\rm w})$ of the *canonical relation* for the Fourier Integral Operator given by (42).

The wavefront set of η_1 is the same as determined for the weak scatterer case:

$$\mathrm{WF}(\eta_1) \subseteq \bigcup_{j=1,2} \left\{(\theta_n, t, \alpha; \sigma'', \tau'', \gamma'' = 0) : \ \alpha^j_{j;n} t_n - (1 + \alpha^j_{j;n}) r_n(\boldsymbol{z}^j)/c = 0, \right.$$
$$\left. (\sigma'', \tau'') = \omega'(1, -\alpha^j_{j;n}) \right\}. \tag{58}$$

The wavefront set of η_2 is

$$\mathrm{WF}(\eta_2) \subseteq \bigcup_{j=1,2} \left\{(\theta_n, t, \alpha; \sigma'', \tau'', \gamma'' = 0) : \ \alpha^{j'}_{j;n} t_n - (r_n(\boldsymbol{z}^{j'}) + \alpha^{j'}_{j;n} r_n(\boldsymbol{z}^j) + L)/c = 0, \right.$$
$$\left. (\sigma'', \tau'') = \omega'(1, -\alpha^{j'}_{j;n}) \right\}. \tag{59}$$

The wavefront set of η_3 is

$$\mathrm{WF}(\eta_3) \subseteq \bigcup_{j=1,2} \left\{(\theta_n, t, \alpha; \sigma'', \tau'', \gamma'' = 0) : \ \alpha^j_{j;n} t_n - ((1 + \alpha^j_{j;n}) r_n(\boldsymbol{z}^j) + 2L)/c = 0, \right.$$
$$\left. (\sigma'', \tau'') = \omega'(1, -\alpha^j_{j;n}) \right\}. \tag{60}$$

Finally, the wavefront set of our three-term approximation to η_{mult} is the union $\mathrm{WF}(\eta_1) \cup \mathrm{WF}(\eta_2) \cup \mathrm{WF}(\eta_3)$.

We note that the critical curves in the θ_n–t_n plane are somewhat different for the single-, double-, and triple-scattering contributions. In particular, single-scattering curves are

$$t_n = 2\frac{R - \mathcal{O}^{\mathrm{T}}(\theta_n)\hat{\boldsymbol{x}} \cdot \boldsymbol{z}}{c + \dot{\mathcal{O}}^{\mathrm{T}}(\theta_n)\hat{\boldsymbol{x}} \cdot \boldsymbol{z}}, \tag{61}$$

double-scattering curves are described by

$$t_n = 2\,\frac{\left(1 + \frac{1}{2c}\dot{\mathcal{O}}^{\mathrm{T}}(\theta_n)\hat{\boldsymbol{x}} \cdot (\boldsymbol{z}' - \boldsymbol{z})\right)\left(R - \mathcal{O}^{\mathrm{T}}(\theta_n)\hat{\boldsymbol{x}} \cdot \boldsymbol{z}'\right) - \frac{1}{2c}L\,\dot{\mathcal{O}}^{\mathrm{T}}(\theta_n)\hat{\boldsymbol{x}} \cdot \boldsymbol{z} + \frac{1}{2}L}{c + \dot{\mathcal{O}}^{\mathrm{T}}(\theta_n)\hat{\boldsymbol{x}} \cdot \boldsymbol{z}'}, \tag{62}$$

and triple-scattering curves obey

$$t_n = 2\frac{R + L - \mathcal{O}^{\mathrm{T}}(\theta_n + L/c)\hat{\boldsymbol{x}} \cdot \boldsymbol{z}}{c + \dot{\mathcal{O}}^{\mathrm{T}}(\theta_n)\hat{\boldsymbol{x}} \cdot \boldsymbol{z}}. \tag{63}$$

Multiple scattering from pairs of scattering centers can potentially be recognized in the data by the occurrence of collections of such curves.

3.4. Wavefront sets for scattering by reentrant structures

The dispersive-scattering model of equation (23) links the downrange artifacts of equation (27) to the target image through the $\varphi_{\mathrm{M}}(\boldsymbol{z})$ factor. We choose φ_{M} by Theorem 1 so that it is supported in a neighborhood of M and its wavefront set is

$$\mathrm{WF}(\varphi_{\mathrm{M}}) = \left\{(\tilde{\boldsymbol{z}}, \tilde{\boldsymbol{\zeta}}) : \tilde{\boldsymbol{z}} \in M, A(\tilde{\boldsymbol{\zeta}} \cdot \hat{\boldsymbol{N}}) \geq 0\right\}. \tag{64}$$

We compute $\mathrm{WF}(\eta_{\mathrm{d}})$ in several steps: a) compute $\mathrm{WF}(q_{\mathrm{M}})$ from Theorem 3; b) compute $\mathrm{WF}(I_m)$; c) compute $\mathrm{WF}(K_{\mathrm{d}})$ from Theorem 2; d) compute $\mathrm{WF}(K_{\mathrm{d}}Q_{\mathrm{d}}) = \mathrm{WF}(K_{\mathrm{d}}I_m q_{\mathrm{M}})$ from Theorem 3; and finally e) consider η_{d} as the push-forward $P_*(K_{\mathrm{d}}Q_{\mathrm{d}})$ for $\mathcal{P} : (\theta_n, t, \alpha, \boldsymbol{z}, t') \mapsto (\theta_n, t, \alpha)$, and compute the wavefront set of η_{d} from Theorem 5.

Step a) The wavefront set of $q_{\mathrm{M}}(\theta_n, \boldsymbol{z}) = A(\hat{\boldsymbol{N}} \cdot \hat{\boldsymbol{R}}_n)\varphi_{\mathrm{M}}(\boldsymbol{z})$ is obtained from Theorem 3:

$$\mathrm{WF}(q_{\mathrm{M}}) \subseteq [\mathrm{WF}(A) + \mathrm{WF}(\varphi_{\mathrm{M}})] \cup \mathrm{WF}(A) \cup \mathrm{WF}(\varphi_{\mathrm{M}})\,. \tag{65}$$

The coupling pattern A, however, is assumed to be smooth; its wavefront set is therefore empty. Consequently, the wavefront set of q_{M} is simply the pull-back of $\mathrm{WF}(\varphi_{\mathrm{M}})$ to $\mathbb{R}^8$ by Theorem 4:

$$\mathrm{WF}(q_{\mathrm{M}}) = \left\{(\tilde{\theta}, \tilde{\boldsymbol{z}}; \tilde{\sigma}, \tilde{\boldsymbol{\zeta}}) : \tilde{\boldsymbol{z}} \in M,\ \tilde{\sigma} = 0,\ A(\tilde{\boldsymbol{\zeta}} \cdot \hat{\boldsymbol{N}}) \geq 0 \text{ and } A(\hat{\boldsymbol{N}} \cdot \mathcal{O}^{\mathrm{T}}(\tilde{\theta})\hat{\boldsymbol{x}}) > 0\right\}. \tag{66}$$

Step b) The wavefront set of I_m can be calculated by cutting out a small interval about $\omega = w_m$ in the definition (28) of f, and then using standard theory [9, 13, 32] to conclude that the wavefront set of I_m is the same as that of HJ_0. Alternatively, one can apply theorems 2, 3, 4, and 5 to draw the same conclusion. Accordingly we have

$$\mathrm{WF}(I_m) = \left\{(\tilde{t}', \tilde{\tau}') : \tilde{t}' = 2L/c, \tilde{\tau}' = a,\ \text{an arbitrary nonzero real number}\right\}\,. \tag{67}$$

We note that this wavefront set is independent of m.

Step c) K_{d} has a phase function depending on the additional variable t':

$$\phi = -\omega'(\alpha_n' t_n - (1 + \alpha_n')r_n/c - t') \tag{68}$$

and so, by Theorem 2,

$$\mathrm{WF}(K_{\mathrm{d}}) \subseteq \Bigg\{(\theta_n, t, \alpha, \boldsymbol{z}, t'; \sigma, \tau, \gamma = 0, \boldsymbol{\zeta}, \tau') :\ \alpha_n'(\boldsymbol{z})t_n - (1 + \alpha_n'(\boldsymbol{z}))r_n(\boldsymbol{z})/c - t' = 0,$$
$$\sigma = \omega',\ \tau = -\omega'\alpha_n'(\boldsymbol{z}),\ \boldsymbol{\zeta} = \frac{2\omega'}{c + \nu_n(\boldsymbol{z})}\hat{\boldsymbol{R}}(\tfrac{1}{2}(t_n + t')),\ \tau' = \omega'\Bigg\}. \tag{69}$$

Step d) As before, we use Theorem 3 to obtain $\mathrm{WF}(K_{\mathrm{d}}Q_{\mathrm{d}}) = \mathrm{WF}(K_{\mathrm{d}}I_m q_{\mathrm{M}})$:

$$\mathrm{WF}(K_{\mathrm{d}}I_m q_{\mathrm{M}}) \subseteq (\mathrm{WF}(K_{\mathrm{d}}) + \mathrm{WF}(I_m) + \mathrm{WF}(q_{\mathrm{M}})) \cup \mathrm{WF}(K_{\mathrm{d}}) \cup \mathrm{WF}(I_m) \cup \mathrm{WF}(q_{\mathrm{M}})\,. \tag{70}$$

The sum term is

$$\mathrm{WF}(K_{\mathrm{d}}) + \mathrm{WF}(I_m) + \mathrm{WF}(q_{\mathrm{M}}) \subseteq \Bigg\{(\theta_n, t, \alpha, \boldsymbol{z}, t'; \check{\sigma}, \check{\tau}, \check{\gamma} = 0, \check{\boldsymbol{\zeta}}, \check{\tau}') : \tag{71}$$
$$\alpha_n'(\boldsymbol{z})t_n - (1 + \alpha_n'(\boldsymbol{z}))r_n(\boldsymbol{z})/c - t' = 0,$$
$$\text{where } \boldsymbol{z} = \tilde{\boldsymbol{z}}, \tilde{\boldsymbol{z}} \in M, t' = \frac{2L}{c},\ \text{and } A(\hat{\boldsymbol{N}} \cdot \mathcal{O}^{\mathrm{T}}(\tilde{\theta})\hat{\boldsymbol{x}}) > 0,$$
$$\check{\sigma} = \omega' + 0 + 0,$$
$$\check{\tau} = -\omega'\alpha_n'(\tilde{\boldsymbol{z}}) + 0 + 0,$$
$$\check{\boldsymbol{\zeta}} = \frac{2\omega'}{c + \nu_n(\boldsymbol{z})}\hat{\boldsymbol{R}}(\tfrac{1}{2}(t_n + t')) + 0 + \tilde{\boldsymbol{\zeta}},\ \text{where } A(\tilde{\boldsymbol{\zeta}} \cdot \hat{\boldsymbol{N}}) \geq 0,$$
$$\check{\tau}' = \omega' + a + 0\Bigg\},$$

where ω' and a are arbitrary nonzero real numbers.

We compute $\mathrm{WF}(K_{\mathrm{d}}I_m q_{\mathrm{M}})$ by (70), where $\mathrm{WF}(K_{\mathrm{d}})$ is given by (69), $\mathrm{WF}(I_m)$ is given by (67) (pulled back to $\mathbb{R}^{14}$ by Theorem 4), and $\mathrm{WF}(q_{\mathrm{M}})$ is given by (66) (pulled back to $\mathbb{R}^{14}$).

Step e) The radar data (30) can be written as the push-forward $\eta_{\rm d} = \mathcal{P}_*(K_{\rm d}Q_{\rm d}) = \mathcal{P}_*(K_{\rm d}I_m q_{\rm M})$ where $\mathcal{P} : (\theta_n, t, \alpha, \boldsymbol{z}, t') \mapsto (\theta_n, t, \alpha)$. The Jacobian of this projection is

$$D\mathcal{P} = (\mathbf{I}_{3\times 3}\mathbf{0}_{3\times 4}) \ . \tag{72}$$

The wavefront set of $\eta_{\rm d}$ can therefore be calculated from Theorem 5:

$$\begin{aligned}\mathrm{WF}(\eta_{\rm d}) = \mathrm{WF}(\mathcal{P}_*(K_{\rm d}I_m q_{\rm M})) \subseteq \Big\{(\theta_n, t, \alpha; \sigma'', \tau'', \gamma'') : \\ \big(\mathcal{P}(\theta_n, t, \alpha, \boldsymbol{z}, t'); (D\mathcal{P})^{\rm T}(\sigma'', \tau'', \gamma'')\big) \in \mathrm{WF}(K_{\rm d}I_m q_{\rm M})\Big\} \ ,\end{aligned} \tag{73}$$

where the wavefront set of $K_{\rm d}I_m q_{\rm M}$ is given by (70). From (72) we see that $(D\mathcal{P})^{\rm T}(\sigma'', \tau'', \gamma'') = (\sigma'', \tau'', \gamma'', \mathbf{0}, 0)$; in other words, we consider elements of (70) for which $\boldsymbol{\zeta} = \mathbf{0}$ and $\tau' = 0$. From these requirements, we see that the elements from $\mathrm{WF}(I_m)$ do not contribute because $\check{\tau}' = a \neq 0$. Similarly, $\mathrm{WF}(K_{\rm d})$ and $\mathrm{WF}(q_{\rm M})$ do not contribute because $\boldsymbol{\zeta} \neq \mathbf{0}$.

Consequently, $\mathrm{WF}(\eta_{\rm d})$ is contained in the set

$$\begin{aligned}\mathrm{WF}(\eta_{\rm d}) \subseteq \Bigg\{(\theta_n, t, \alpha; \sigma'', \tau'', \gamma'' = 0) : \ \alpha_n'(\boldsymbol{z})t_n - ((1+\alpha_n'(\boldsymbol{z}))r_n(\boldsymbol{z}) + 2L)/c = 0, \\ \text{where } \boldsymbol{z} \in M, \ A(\hat{\boldsymbol{N}} \cdot \mathcal{O}^{\rm T}(\tilde{\theta})\hat{\boldsymbol{x}}) > 0, \ \text{and } A\left(\hat{\boldsymbol{R}}(\tfrac{1}{2}t_n + L/c) \cdot \hat{\boldsymbol{N}}\right) \geq 0, \\ (\sigma'', \tau'') = \omega'(1, -\alpha_n'(\boldsymbol{z}))\Bigg\} \ .\end{aligned} \tag{74}$$

We see that the critical curve in the θ_n–t_n plane is the same as (63), and is associated with scattering centers lying within the duct/cavity at distance L from the mouth. The point $\boldsymbol{z}$ in the critical set corresponds to a point at the mouth of the reentrant structure. In addition, the critical curve is present in the data only at angles for which energy couples into the dispersive structure, and for times after which the wave has reached the scattering center within.

4. Examples and interpretation

4.1. Ordinary ISAR

"Ordinary ISAR" [3, 4, 28] considers the target to be composed entirely of weak scatterers and is described by the discussion of section 2.1. In addition, ISAR traditionally considers the target to be rotating at a constant rate about a fixed axis, and that this rate is sufficiently slow to ensure that $\nu_n(\boldsymbol{z}) \ll c$ for all $\boldsymbol{z}$ in the target's support. (More accurately, traditional ISAR is usually *constrained* to use data collected over a sufficiently small time interval that a constant rotation rate can be considered a good approximation.) In practice, data collection times are on the order of seconds, so that typical apertures are several degrees.

To illustrate the ideas, we assume that the target is rotating at a constant rate Ω about the axis $(0, 0, 1)$, so that

$$\mathcal{O}(t) = \begin{pmatrix} \cos\Omega t & \sin\Omega t & 0 \\ -\sin\Omega t & \cos\Omega t & 0 \\ 0 & 0 & 1 \end{pmatrix} \ . \tag{75}$$

Further, we assume that the radar is located in the $\hat{\boldsymbol{x}} = (1, 0, 0)^{\rm T}$ direction. In this case, we can write $r_n(\boldsymbol{z}) = R - \boldsymbol{z}_\perp \cdot \hat{\boldsymbol{\mu}}(\Omega\theta_n)$, where $\boldsymbol{z}_\perp = (z_1, z_2)$ and $\hat{\boldsymbol{\mu}}(\Omega\theta_n) = (\cos\Omega\theta_n, \sin\Omega\theta_n)$. Substituting (75) into (15), performing the integration over the rotation

axis, writing $Q_{\rm sum}(\boldsymbol{z}_\perp) \equiv \int Q_{\rm w}(\boldsymbol{z}){\rm d}z_3$, and dropping terms in $\omega'(\nu/c)^2$ in the phase and ν/c in the modulus yields

$$\eta_{\rm w}(\theta_n, t_n, \alpha) \approx \frac{1}{(4\pi R)^2\alpha}\int Q_{\rm sum}(\boldsymbol{z}_\perp)\,{\rm e}^{-{\rm i}[(\omega'+\omega'_{{\rm D};n}(\boldsymbol{z}_\perp))t_n-(2\omega'+\omega'_{{\rm D};n}(\boldsymbol{z}_\perp))(R-\boldsymbol{z}_\perp\cdot\hat{\boldsymbol{\mu}}(\Omega\theta_n))/c]}$$
$$\times\,({\rm i}\omega')^2 S_{\rm inc}(\omega')\overline{S}_{\rm inc}\left(\alpha^{-1}(\omega'+\omega'_{{\rm D};n})\right)\,{\rm d}\omega'{\rm d}\boldsymbol{z}_\perp\,, \tag{76}$$

where we have written $\omega'\alpha'(\boldsymbol{z}_\perp) = \omega' + \omega_{{\rm D};n}(\boldsymbol{z}_\perp)$, with $\omega_{{\rm D};n}(\boldsymbol{z}_\perp) = -2\nu_n(\boldsymbol{z}_\perp)\omega'/c$ denoting the Doppler frequency *shift*. We note that dropping terms of order $\omega(\nu/c)^2$ and higher involves a *narrowband* assumption [3, 8], i.e., $S_{\rm inc}$ is negligible outside an interval (ω_1, ω_2) with $\omega_2 - \omega_1 \ll \omega_1$.

Equation (76) demonstrates that standard ISAR imaging can only recover the axis-integrated target scattering density function. In high range resolution (HRR) ISAR imaging, equation (76) can be further reduced since such systems use $s_{\rm inc}(t)$ that will best approximate equation (17) as $\mathcal{A}(\alpha(t-t'),\alpha) \approx \delta(t-t')$. For HRR signals, $S_{\rm inc}(\omega)$ will be a slowly varying function of ω and $S_{\rm inc}(\omega)\overline{S}_{\rm inc}\left(\alpha^{-1}(\omega+\delta\omega)\right)$ will also be slowly varying when $|\delta\omega| \ll |\omega|$. Consequently, HRR systems will have poor frequency resolution capabilities and $({\rm i}\omega')^2 S_{\rm inc}(\omega')\overline{S}_{\rm inc}\left(\alpha^{-1}(\omega'+\omega'_{{\rm D};n})\right)$ is often approximated as being constant over ${\rm supp}\, S_{\rm inc} = (\omega_1, \omega_2)$. In this case, equation (76) reduces to

$$\eta_{\rm w}(\theta_n, t_n) \propto \iint_{\omega_1}^{\omega_2} Q_{\rm sum}(\boldsymbol{z}_\perp)\,{\rm e}^{-{\rm i}[(\omega'+\omega'_{{\rm D};n}(\boldsymbol{z}_\perp))t_n-(2\omega'+\omega'_{{\rm D};n}(\boldsymbol{z}_\perp))(R-\boldsymbol{z}_\perp\cdot\hat{\boldsymbol{\mu}}(\Omega\theta_n))/c]}{\rm d}\omega'{\rm d}\boldsymbol{z}_\perp\,, \tag{77}$$

where we have suppressed the (now formal) α-dependence on the left side because the right side is independent of α.

Systems with coarse frequency resolution are also insensitive to local Doppler shift $\omega_{{\rm D};n}(\boldsymbol{z}_\perp)$. These frequency shifts are actually quite small; typical values for maneuvering aircraft targets are $\omega_{{\rm D};n}/\omega < 10^{-7}$ and we can make the approximation $\omega_{{\rm D};n}/\omega \ll 1$. In other words, we assume that the target is effectively stationary for the duration of the fast-time measurement. This simplification is known as the *start-stop* approximation. Such measurements are Doppler-free and, in this case, the term "range-Doppler" imaging is something of a misnomer—the methods used are really closer to tomographic techniques. When $\omega_{{\rm D};n} = 0$, equation (77) becomes

$$\eta_{\rm w}(\theta_n, t_n) \propto \int Q_{\rm sum}(\boldsymbol{z}_\perp)\int_{\omega_1}^{\omega_2}{\rm e}^{-{\rm i}\omega'[t_n-2(R-\boldsymbol{z}_\perp\cdot\hat{\boldsymbol{\mu}}(\Omega\theta_n))/c]}{\rm d}\omega'{\rm d}\boldsymbol{z}_\perp\,. \tag{78}$$

We see that if $\omega_1 \to -\infty$ and $\omega_2 \to \infty$, then equation (78) is precisely a Radon transform of $Q_{\rm sum}$: for each θ_n, one integrates $Q_{\rm sum}$ over the line $t_n = 2(R - \boldsymbol{z}_\perp\cdot\hat{\boldsymbol{\mu}}(\Omega\theta_n))/c$. For real systems, (78) is a bandlimited version of the Radon transform and ISAR images are traditionally produced by Radon inversion methods (such as filtered backprojection [20, 21]).

While ISAR *imaging* schemes are usually based on equation (78), the analysis of section 3 shows that the wavefront set of the data contains considerable information about the target, which can be extracted without forming an image. The wavefront analysis of section 3.2 simplifies somewhat with the start-stop approximation. In particular, since $\dot{\mathcal{O}}(t) = 0$ during the fast time interval, $\alpha'_n = 1$ and we can take $\hat{\boldsymbol{R}}(\frac{1}{2}t_n) = \hat{\boldsymbol{R}}(\theta_n)$ in the expression for $\boldsymbol{\zeta}$ of (55). Moreover, for (78) we have ${\rm d}\hat{\boldsymbol{R}}/{\rm d}\theta_n = \hat{\boldsymbol{\mu}}^\perp(\theta_n) = (\sin\theta_n, -\cos\theta_n)$, which implies that (55) reduces to

$$\mathrm{WF}(\eta_{\rm w}) \subseteq \Big\{(\theta_n, t_n; \sigma, \tau) : t_n \approx 2r_n(\tilde{\boldsymbol{z}}_\perp)/c, \quad (\sigma,\tau) = -\omega\left(2\tilde{\boldsymbol{z}}_\perp\cdot\hat{\boldsymbol{\mu}}^\perp(\theta_n)/c, -1\right),$$
$$\text{where } \tilde{\boldsymbol{\zeta}}_\perp = -2\omega\hat{\boldsymbol{R}}(\theta_n)/c \text{ for some } (\tilde{\boldsymbol{z}}_\perp, \tilde{\boldsymbol{\zeta}}_\perp) \in \mathrm{WF}(Q_{\rm w})\Big\}\,. \tag{79}$$

It is easy to see that $\tau = -\omega(1+2\tilde{\boldsymbol{z}}_\perp\cdot\hat{\boldsymbol{\mu}}^\perp/c) = -\omega(1+2\nu(\boldsymbol{z}_\mathrm{T})/c) = -(\omega+\omega_D)$ encodes an *inferred* Doppler shift across the synthetic aperture collection interval (even though no local frequency shifts are measured).

A point scatterer (whose wavefront set contains all directions $\boldsymbol{\xi}$) located at $\boldsymbol{z}$ corresponds to the curve $ct_n = 2R - 2\boldsymbol{z}_\perp\cdot\hat{\boldsymbol{\mu}}(\Omega\theta_n)$ in the data domain. The coordinates of this scatterer are usually estimated from the intersection of the backprojections constructed from data (i.e., lines oriented with angle $\Omega\theta_n$ and offset $2\boldsymbol{z}_\perp\cdot\hat{\boldsymbol{\mu}}(\Omega\theta_n)$). But the wavefront-set analysis suggests another possibility: find the range $2\boldsymbol{z}_\perp\cdot\hat{\boldsymbol{\mu}}(\Omega\theta_n)$ from knowledge of t_n and estimate the cross-range position from the directions $(\sigma,\tau) = -\omega(2\tilde{\boldsymbol{z}}_\perp\cdot\hat{\boldsymbol{\mu}}^\perp/c, -1)$.

Strictly speaking, of course, bandlimited data are smooth and therefore the wavefront set is empty. Our analysis, however, views the bandlimited case as an approximation to the infinite-bandwidth problem.

4.2. High Doppler resolution

In the situation where the incident signal is chosen so that $S_\mathrm{inc}(\omega') \sim \delta(\omega'-\omega_0)$, the radar becomes a *high Doppler resolution* system [26]. Of course, in this situation we cannot actually correlate with $s_\mathrm{inc}(\alpha t)$ since such a signal would be infinitely long. Instead, we set $\theta_1 = 0$ and consider a "long-duration pulse" approximation to $\mathcal{F}\{\overline{s}_\mathrm{inc}\}(\omega'')$:

$$\overline{S}_\mathrm{corr}(\omega'') = \int_{-T}^{T} \mathrm{e}^{-\mathrm{i}(\omega''-\omega_0)t'}\,\mathrm{d}t'\,, \tag{80}$$

where T is "large." Substitution into equation (12) yields

$$\begin{aligned}
\eta_\mathrm{D}(t,\alpha) &= \frac{1}{32\pi^3R^2}\iint_{-T}^{T} Q_\mathrm{w}(\boldsymbol{z})\,(\mathrm{i}\omega')^2\delta(\omega'-\omega_0)\mathrm{e}^{-\mathrm{i}(\omega''-\omega_0)t'}\,\mathrm{e}^{-\mathrm{i}\omega'[t''-(R+\hat{\boldsymbol{R}}(t'')\cdot\boldsymbol{z})/c]} \\
&\qquad\qquad \times\,\mathrm{e}^{-\mathrm{i}\omega''\alpha(t-\theta_n-t''-(R+\hat{\boldsymbol{R}}(t'')\cdot\boldsymbol{z})/c]}\,\mathrm{d}t'\mathrm{d}\omega'\mathrm{d}\omega''\mathrm{d}t''\mathrm{d}\boldsymbol{z} \\
&= \frac{-\omega_0^2}{(4\pi)^2R^2}\iint_{-T}^{T} Q_\mathrm{w}(\boldsymbol{z})\,\mathrm{e}^{-\mathrm{i}\omega_0(t''-t'-(R+\hat{\boldsymbol{R}}(t'')\cdot\boldsymbol{z})/c)} \\
&\qquad\qquad \times\,\delta(t'+\alpha(t-t''-(R+\hat{\boldsymbol{R}}(t'')\cdot\boldsymbol{z})/c))\,\mathrm{d}t'\mathrm{d}t''\mathrm{d}\boldsymbol{z} \\
&= \frac{-\omega_0^2}{(4\pi)^2R^2}\int_{\mathcal{T}} Q_\mathrm{w}(\boldsymbol{z})\,\mathrm{e}^{-\mathrm{i}\omega_0(\alpha t+(1-\alpha)t''-(1+\alpha)\hat{\boldsymbol{R}}(t'')\cdot\boldsymbol{z}/c)}\,\mathrm{d}t''\mathrm{d}\boldsymbol{z}\,.
\end{aligned} \tag{81}$$

The set $\mathcal{T} = \{(t'',\boldsymbol{z}) : -T/\alpha < t-t''-(R+\hat{\boldsymbol{R}}(t'')\cdot\boldsymbol{z})/c < T\alpha\}$ in the last equation of (81) consists of those values of t'' and $\boldsymbol{z}$ for which the argument of the delta function in the previous line can vanish. When $R \gg \hat{\boldsymbol{R}}(t'')\cdot\boldsymbol{z}$, we can use the fact that high Doppler resolution signals yield coarse time resolution to approximate $\mathcal{T}$ by $\mathcal{T} \approx \{t'' : -T/\alpha < t-t''-R/c < T/\alpha\}$. In equation (81) we then make the change of variables $t' = t''-(t-R/c)$ to obtain

$$\eta_\mathrm{D}(t,\alpha) \approx \frac{-\omega_0^2\,\mathrm{e}^{-\mathrm{i}\omega_0[t-(1-\alpha)R/c]}}{(4\pi)^2R^2}\iint_{-T}^{T} Q_\mathrm{w}(\boldsymbol{z})\mathrm{e}^{-\mathrm{i}\omega_0[(1-\alpha)t'-(1+\alpha)\hat{\boldsymbol{R}}(t'+t-R/c)\cdot\boldsymbol{z}/c]}\,\mathrm{d}t'\mathrm{d}\boldsymbol{z}\,. \tag{82}$$

We would like to carry out a wavefront-set analysis for (82) as we did in section 3. Here the time variable t' should correspond to the frequency variable ω in section 3. Unfortunately, for rotating targets, the phase of (82) is not generally homogeneous of degree one in the integration variable t', and a more traditional approach is required (e.g., the method of stationary phase).

We can apply the wavefront-set analysis of section 3, however, when the rotation rate Ω is small. In this case, we use the fact that for rotations of the form (75), we can write $\mathcal{O}(t'+t-R/c)=\mathcal{O}(t-R/c)\mathcal{O}(t')$, so that $\hat{\boldsymbol{R}}(t'+t-R/c)=\mathcal{O}(t')\hat{\boldsymbol{R}}(t-R/c)$. When Ω is sufficiently slow that the small-angle approximation can be used over the whole data collection interval, i.e., $\cos\Omega t'\approx 1$ and $\sin\Omega t'\approx\Omega t'$ for all t' in the interval $[-T,T]$, then (82) becomes

$$\eta_{\mathrm{D}}(t,\alpha)\approx\frac{-\omega_0^2\,\mathrm{e}^{-\mathrm{i}\omega_0[t-(1-\alpha)R/c]}}{(4\pi)^2R^2}\iint_{-T}^{T}Q_{\mathrm{sum}}(\boldsymbol{z}_\perp)\mathrm{e}^{-\mathrm{i}\omega_0c^{-1}(1+\alpha)\boldsymbol{z}_\perp\cdot\hat{\boldsymbol{\mu}}(\Omega(t-R/c))}\times\mathrm{e}^{-\mathrm{i}\omega_0t'[1-\alpha-\Omega c^{-1}(1+\alpha)\boldsymbol{z}_\perp\cdot\hat{\boldsymbol{\mu}}^\perp(\Omega(t-R/c))]}\,\mathrm{d}t'\mathrm{d}\boldsymbol{z}_\perp\,, \tag{83}$$

where, as before, $\hat{\boldsymbol{\mu}}^\perp(t)=(\sin t,-\cos t)$. The wavefront set of η_{D} can be calculated in the same manner as done in section 3.2, where ω is replaced by t' and Q_{w} is replaced by $Q_{\mathrm{sum}}(\boldsymbol{z}_\perp)\exp[-\mathrm{i}\omega_0c^{-1}(1+\alpha)\boldsymbol{z}_\perp\cdot\hat{\boldsymbol{\mu}}(\Omega(t-R/c))]$. In this case the phase is

$$\phi=-\omega_0[1-\alpha-\Omega c^{-1}(1+\alpha)\boldsymbol{z}_\perp\cdot\hat{\boldsymbol{\mu}}^\perp(\Omega(t-R/c))]\,. \tag{84}$$

For the wavefront set we obtain:

$$\begin{aligned}\mathrm{WF}(\eta_{\mathrm{D}})\subseteq\Bigg\{(t,\alpha;\tau,\gamma):\ &\frac{1-\alpha}{1+\alpha}=\frac{\Omega}{c}\boldsymbol{z}_\perp\cdot\hat{\boldsymbol{\mu}}^\perp(\Omega(t-R/c))\,,\\ &\tau=\partial_t\phi=-\omega_0\Omega^2c^{-1}(1+\alpha)\boldsymbol{z}_\perp\cdot\hat{\boldsymbol{\mu}}(\Omega(t-R/c))\,,\\ &\gamma=\partial_\alpha\phi=t'\omega_0[1+\Omega c^{-1}\boldsymbol{z}_\perp\cdot\hat{\boldsymbol{\mu}}^\perp(\Omega(t-R/c))\\ &\qquad=\tfrac{1}{2}t'[2\omega_0+\omega_{\mathrm{D}}(\boldsymbol{z},t-R/c)]\,,\\ &\boldsymbol{\zeta}_{\mathrm{T}}=-\Omega c^{-1}(1+\alpha)\hat{\boldsymbol{\mu}}^\perp(\Omega(t-R/c))\\ &\text{for some }(\boldsymbol{z}_\perp,\boldsymbol{\zeta}_\perp)\in\mathrm{WF}(Q_{\mathrm{sum}})\Bigg\}\,.\end{aligned} \tag{85}$$

We see that knowledge of the wavefront set again enables us to estimate the location of point scatterers: the critical curve (or γ) gives us $\boldsymbol{z}_\perp\cdot\hat{\boldsymbol{\mu}}^\perp$, and knowledge of τ gives us $\boldsymbol{z}_\perp\cdot\hat{\boldsymbol{\mu}}$.

4.3. Accelerating targets

The case where $|\ddot{\mathcal{O}}|$ is not vanishingly small can be even more difficult. In this situation, the approximation $\mathcal{O}(t''+\theta_n)\approx\mathcal{O}(\theta_n)+\dot{\mathcal{O}}(\theta_n)t''$ may not be valid and, consequently, equation (12) cannot be simplified to the form of equation (14). The manifestation of this failure is progressive phase error in η and degradation of ISAR images constructed under the assumption that Ω is constant—such images are said to be "de-focused."

When $\ddot{\mathcal{O}}(t)$ is smooth and there are no abrupt changes in $\dot{\mathcal{O}}(t)$, then it is possible to consider these data as having been collected over a collection of subintervals $(\theta_{N_1},\theta_{N_2})$ whose lengths are sufficiently small so that $\frac{1}{2}(t'')^2\ddot{\mathcal{O}}(\theta_n)\approx 0$ for $\theta_n\in(\theta_{N_1},\theta_{N_2})$. Over each such subinterval, $\dot{\mathcal{O}}$ can be considered to be constant in a manner similar to the start-stop approximation discussed in section 4.1. Within each subinterval the analysis of this paper can be applied; but over the collection of subintervals the critical points will not form sine curves. The normal directions $(\sigma,\tau)=(\partial_{t_n}\phi,\partial_{\theta_n}\phi)$ track these non-sinusoidal variations in HRR data.

In the high-Doppler-resolution case, the small-angle expedient used in section 4.2 (approximating the phase as a degree-one homogeneous function of t') will generally fail to accurately capture the aspect dependence of data collected from accelerating targets. The usual plan of attack for this problem is to identify $(-T,T)$ with $(\theta_{N_1},\theta_{N_2})$ and choose time-domain signals that are short enough for Ω to be considered approximately constant on

$(\theta_{N_1}, \theta_{N_2})$ but are also long enough to enable good estimation of the instantaneous frequency (for retrieval of $\Omega(\theta_n)$). These considerations are the same as those normally encountered in time-frequency signal processing [7].

Alternatively, it is possible that the analysis of this paper could be extended to treat the accelerating-target case; this we leave for future work.

5. Conclusions and Future Work

Our discussion has not actually been about radar imaging. Instead, it has focused on the structure imposed upon measured radar data by a class of image features associated with the singular set of the radar target. Standard radar imaging schemes attempt to estimate precisely this class of features, however, and so our approach has "imaging" at its heart. In particular, we have shown that when the weak-scattering approximation is valid, the location of the target's scattering centers can be estimated directly from the data wavefront set.

We have also shown that the mapping from target to data for the important cases of structural dispersion and multiple scattering displays fundamental differences from the weak-scatterer case. In particular, we demonstrated that the wavefront set for multiple-scattering events can be distinguished from single-scattering data. We also showed that the wavefront set for scattering from ducts and cavities is similar to that of a triply-scattered wave. Both these observations are potentially significant: they may lead to schemes for eliminating ISAR image artifacts by first isolating the wavefront set of the measured data and then constructing an image from this reduced data set (this is an area of future research).

The case of three-dimensional ISAR imaging—which relies on a more general version of $\mathcal{O}(t)$ than considered in equation (75)—also fits neatly into this framework. For non-cooperative targets, a principal problem lies in discovering the various roll/pitch/yaw data variations due to target maneuvers from the data themselves. These dependencies must be separately isolated if an accurate image is to be formed, and wavefront-set analysis offers a systematic approach for investigating the target behavior.

We leave for the future the question of how knowledge of the singular structure of the radar data can best be exploited for target imaging and identification. There are a number of issues here. For image formation, the wavefront-set analysis suggests that reconstruction methods related to local tomography [11, 16] may be useful. In particular, analysis of wavefront sets can determine whether backprojection will provide an image free of certain artifacts [22, 24]. In addition, wavefront-set analysis suggests an approach for producing artifact-free, superresolved images: remove all components of the data set except those that correspond to well-understood target features, and form an image from those components only.

Practical implementation of the analysis in this paper requires that we be able to extract the wavefront set from noisy, band-limited and discretely sampled radar data. The problem of extracting wavefront sets under such conditions is closely related to image processing problems such as edge detection, and these are active areas of current research. We explore one possible approach in [5], where we provide numerical examples of synthetic radar data and show how the wavefront set analysis enables us to estimate target parameters from very noisy data.

Our ultimate goal, of course, is to identify targets under all weather conditions. For target identification, it may not be necessary to form an image. We have shown that wavefront set analysis distills the data set into a set of primitives that are easily related to a target's structural properties. As such, this approach constitutes a promising new tool that deserves further investigation.

Acknowledgments

We are grateful to the Mathematical Sciences Research Institute and in particular to the organizers of the fall 2001 program on Inverse Problems for bringing us together; it was there that we began the discussions that led to this work. We also thank Rensselaer Polytechnic Institute and the Naval Air Warfare Center Weapons Division for making possible M.C.'s visit to China Lake during the spring of 2002. M.C. is also grateful to George Papanicolaou, Air Force Office of Scientific Research grant F49620-01-1-0465, and Stanford University for hospitality during the spring of 2002, when much of the paper was written. We would like to thank Rafe Mazzeo for pointing out Theorem 1 to us.

This work was supported by the Office of Naval Research. This work was also supported in part by CenSSIS, the Center for Subsurface Sensing and Imaging Systems, under the Engineering Research Centers Program of the National Science Foundation (award number EEC-9986821), and by the NSF Focused Research Groups in the Mathematical Sciences program.

Appendix A. Multiple Scattering

For a time-harmonic incident wave $U^{\rm inc}(\boldsymbol{x})$, the frequency-domain field $U^{\rm sc}$ scattered from N "point" scatterers can be obtained from the Foldy-Lax [33] equations together with the assumption that the scattered field from a single "point" scatterer is proportional to the Green's function G [27]:

$$U^{\rm sc}(\boldsymbol{x}) = \sum_{j=1}^{N} G(|\boldsymbol{x}-\boldsymbol{z}^j|)\mu_j U_j(\boldsymbol{z}^j) \tag{A.1}$$

$$U_j(\boldsymbol{x}) \;= U^{\rm inc}(\boldsymbol{x}) + \sum_{i\neq j} G(|\boldsymbol{x}-\boldsymbol{z}^i|)\mu_i U_i(\boldsymbol{z}^i), \quad j=1,2,\ldots,N, \tag{A.2}$$

where $G(r) = (4\pi r)^{-1}\exp({\rm i}\omega r/c)$. Equation (A.1) says that the scattered field is the sum of the fields scattered from each scatterer; moreover, the field scattered from the jth scatterer is proportional to the field U_j that is incident upon the jth scatterer. Equations (A.2) say that the jth local incident field is the overall incident field plus the field scattered from all the other scatterers. If the scattering strengths $\mu_1, \mu_2, \ldots, \mu_N$ are known, the equations (A.2) can be solved for the U_j; then the total field can be found from (A.1).

In the case of two "point" scatterers, equations (A.2) are

$$U_1(\boldsymbol{x}) = U^{\rm inc}(\boldsymbol{x}) + G(|\boldsymbol{x}-\boldsymbol{z}^1|)\mu_2 U_2(\boldsymbol{z}^2) \tag{A.3}$$

$$U_2(\boldsymbol{x}) = U^{\rm inc}(\boldsymbol{x}) + G(|\boldsymbol{x}-\boldsymbol{z}^2|)\mu_1 U_1(\boldsymbol{z}^1) \tag{A.4}$$

Evaluating (A.3) at $\boldsymbol{z}^1$ and (A.4) at $\boldsymbol{z}^2$ gives rise to the system of equations

$$\begin{pmatrix} 1 & -\mu_2 G(L) \\ -\mu_1 G(L) & 1 \end{pmatrix} \begin{pmatrix} U_1(\boldsymbol{z}^1) \\ U_2(\boldsymbol{z}^2) \end{pmatrix} = \begin{pmatrix} U^{\rm inc}(\boldsymbol{z}^1) \\ U^{\rm inc}(\boldsymbol{z}^2) \end{pmatrix}, \tag{A.5}$$

where $L = |\boldsymbol{z}^2 - \boldsymbol{z}^1|$. These equations have the solutions

$$U_j(\boldsymbol{z}^j) = \frac{U^{\rm inc}(\boldsymbol{z}^j) + \mu_{j'} G(L) U^{\rm inc}(\boldsymbol{z}^{j'})}{1-\mu_1\mu_2 G^2(L)}, \qquad j=1,2, \tag{A.6}$$

where $j'=2$ if $j=1$ and $j'=1$ if $j=2$. Using (A.6) in (A.1) yields

$$U^{\rm sc}(\boldsymbol{x}) = \sum_{j=1}^{N} G(|\boldsymbol{x}-\boldsymbol{z}^j|)\mu_j \frac{U^{\rm inc}(\boldsymbol{z}^j) + \mu_{j'} G(L) U^{\rm inc}(\boldsymbol{z}^{j'})}{1-\mu_1\mu_2 G^2(L)}. \tag{A.7}$$

The time-domain scattered field due to the incident field (6) can be found by taking $G(|\boldsymbol{x}-\boldsymbol{z}^j|) = \int \exp(\mathrm{i}\omega'(t-t')g(t-t',|\boldsymbol{x}-\boldsymbol{z}^j|)\,\mathrm{d}t'$ and $U^{\mathrm{inc}}(\boldsymbol{z}^j) = S_{\mathrm{inc}}(\omega')(8\pi^2|\boldsymbol{x}-\boldsymbol{z}^j|)^{-1}\exp(\mathrm{i}\omega'|\boldsymbol{x}-\boldsymbol{z}^j|/c)\exp(\mathrm{i}\omega'\theta_n)$ in (A.7) and Fourier transforming from ω' to t. The exponentials involving t cancel, and we obtain

$$u^{\mathrm{sc}}(t,\boldsymbol{x}) = \sum_{j=1}^{N}\int g(t-t',|\boldsymbol{x}-\boldsymbol{z}^j|)\frac{\mu_j S_{\mathrm{inc}}(\omega')}{1-\mu_1\mu_2\,\mathrm{e}^{\mathrm{i}2\omega' L}/(4\pi L)^2} \times\left(\frac{\mathrm{e}^{\mathrm{i}\omega'|\boldsymbol{x}-\boldsymbol{z}^j|/c}}{8\pi^2|\boldsymbol{x}-\boldsymbol{z}^j|}+\mu_{j'}\frac{\mathrm{e}^{\mathrm{i}2\omega' L}}{4\pi L}\frac{\mathrm{e}^{\mathrm{i}\omega'|\boldsymbol{x}-\boldsymbol{z}^{j'}|/c}}{8\pi^2|\boldsymbol{x}-\boldsymbol{z}^{j'}|}\right)\mathrm{e}^{-\mathrm{i}\omega'(t'-\theta_n)}\,\mathrm{d}\omega'\mathrm{d}t'. \quad \text{(A.8)}$$

Appendix B. Some Formulas

$$\frac{\partial}{\partial\theta_n}\left(\hat{\boldsymbol{R}}_n\cdot\boldsymbol{z}\right) = -\dot{\mathcal{O}}^{\mathrm{T}}(\theta_n)\hat{\boldsymbol{x}}\cdot\boldsymbol{z} = \nu_n(\boldsymbol{z}) \quad \text{(B.1)}$$

$$\frac{\partial\alpha_n}{\partial\nu_n(\boldsymbol{z})} = \frac{-2}{c(1+\nu_n(\boldsymbol{z})/c)^2} \quad \text{(B.2)}$$

$$\frac{\partial\nu_n(\boldsymbol{z})}{\partial\theta_n} = -\ddot{\mathcal{O}}^{\mathrm{T}}(\theta_n)\hat{\boldsymbol{x}}\cdot\boldsymbol{z} \quad \text{(B.3)}$$

$$\nabla_{\boldsymbol{z}}\nu_n(\boldsymbol{z}) = -\dot{\mathcal{O}}^{\mathrm{T}}(\theta_n)\hat{\boldsymbol{x}} \quad \text{(B.4)}$$

$$\nabla_{\boldsymbol{z}}r_n = -\nabla_{\boldsymbol{z}}(\mathcal{O}^{\mathrm{T}}(\theta_n)\hat{\boldsymbol{x}}\cdot\boldsymbol{z}) = -\mathcal{O}^{\mathrm{T}}(\theta_n)\hat{\boldsymbol{x}} \quad \text{(B.5)}$$

$$\nabla_{\boldsymbol{z}}\alpha_n = \frac{\partial\alpha_n}{\partial\nu_n(z)}\nabla_{\boldsymbol{z}}\nu_n(\boldsymbol{z}) = \frac{2\dot{\mathcal{O}}^{\mathrm{T}}(\theta_n)\hat{\boldsymbol{x}}}{c(1+\nu_n(\boldsymbol{z})/c)^2} \quad \text{(B.6)}$$

References

[1] Beylkin G 1985 Imaging of discontinuities in the inverse scattering problem by inversion of a causal generalized Radon transform *J. Math. Phys.* **26** 99–108

[2] Beylkin G and Burridge R 1990 Linearized inverse scattering problems in acoustics and elasticity *Wave Motion* **12** 15–52

[3] Borden B 1999 *Radar imaging of Airborne Targets* (Bristol: Institute of Physics)

[4] Borden B 2002 Mathematical problems in radar inverse scattering *Inverse Problems* **18** R1–R28

[5] Borden B and Cheney M Microlocal ISAR for low signal-to-noise environments *IEEE Trans. on Image Proc.* submitted for review

[6] Bleistein N, Cohen J K, and Stockwell J W 2000 *The Mathematics of Multidimensional Seismic Inversion* (New York: Springer)

[7] Chen V C and Ling H 2002 *Time-Frequency Transforms for Radar Imaging and Signal Analysis* (Boston: Artech House)

[8] Cook C E and Bernfeld M 1967 *Radar Signals* (New York: Academic)

[9] Duistermaat J J 1996 *Fourier Integral Operators* (Boston: Birkhauser)

[10] DeHoop M and Stolk C C 2002 Microlocal analysis of seismic inverse scattering in anisotropic, elastic media *Comm. Pure Appl. Math.* **55** 261–301

[11] Faridani A, Finch D V, Ritman E L, and Smith K T 1997 Local Tomography II *SIAM J. Appl. Math.* **57** 1095–127

[12] Greenleaf A and Uhlmann G 1989 Nonlocal inversion formulas for the X-ray transform *Duke Math. Journal* **58** 205–40

[13] Grigis A and Sjöstrand J 1994 *Microlocal Analysis for Differential Operators: An Introduction (London Mathematical Society Lecture Note Series vol 196)* (Cambridge: Cambridge University Press)

[14] Hörmander L 1983, *The Analysis of Linear Partial Differential Operators I* (New York: Springer)

[15] Herman G T, Tuy H K, Langenberg K J and Sabatier P C 1988 *Basic Methods of Tomography and Inverse Problems* (Philadelphia: Adam Hilger)

[16] Katsevich A I 1997 Local tomography for the generalized Radon transform *SIAM J. App. Math.* **57** 1128–62

[17] Langenberg K J, Brandfass M, Mayer K, Kreutter T, Brüll A, Felinger P and Huo D 1993 Principles of microwave imaging and inverse scattering *EARSeL Adv. Remote Sens.* **2** 163–86
[18] Louis A and Quinto E T 2000 Local tomographic methods in SONAR *Surveys on Solution Methods for Inverse Problems* ed D Colton, H W Engl, A K Louis, J R McLaughlin and W Rundell (New York: Springer)
[19] Mensa D L 1981 *High Resolution Radar Imaging* (Dedham, MA: Artech House)
[20] Natterer F 1986 *The Mathematics of Computerized Tomography* (New York: Wiley)
[21] Natterer F and Wübbeling F 2001 *Mathematical Methods in Image Reconstruction* (Philadelphia: SIAM)
[22] Nolan C J 2000 Scattering near a fold caustic *SIAM J. App. Math.* **61** 659–72
[23] Nolan C J and Cheney M 2002 Synthetic aperture inversion *Inverse Problems* **18** 221–36
[24] Nolan C J and Symes W W 1997 Global solution of a linearized inverse problem for the acoustic wave equation *Comm. Part. Diff. Eq.* **22**
[25] Quinto E T 1993 Singularities of the X-ray transform and limited data tomography in $\mathbb{R}^2$ and $\mathbb{R}^3$ *SIAM J. Math. Anal.* **24** 1215–25
[26] Skolnik M I 1980 *Introduction to Radar Systems* (New York: McGraw-Hill)
[27] Snieder R K and Scales J A 1998 Time-reversed imaging as a diagnostic of wave and particle chaos *Phys. Rev. E* **58** 5668–75
[28] Son J S, Thomas G and Flores B C 2001 *Range-Doppler Radar Imaging ad Motion Compensation* (Boston: Artech House)
[29] Sullivan R J 2000 *Microwave Radar Imaging and Advanced Concepts* (Boston: Artech House)
[30] Taylor M 1981 *Pseudodifferential Operators* (Princeton: Princeton University Press)
[31] Treves F 1975 *Basic Linear Partial Differential Equations* (New York: Academic)
[32] Treves F 1980 *Introduction to Pseudodifferential and Fourier Integral Operators* vol 1 and 2 (New York: Plenum)
[33] Tsang L, Kong J A and Shin R T 1985 *Theory of Microwave Remote Sensing* (New York: Wiley)