

## Notation: What The Heck Stuff Means

- $\frac{dy}{dx}$ 
  - It is not a fraction. It is the derivative of  $y$  with respect to  $x$ . Here  $y$  needs to be a function of only  $x$  (at least in this class). If  $y = f(x)$ , then  $\frac{dy}{dx} = f'(x)$ .
- $\frac{d}{dx}$ 
  - This means ‘the derivative with respect to  $x$  of \_\_\_\_\_’. Currently this is like a sentence that hasn’t been finished. Technically, it is a differential operator; it takes derivatives (with respect to  $x$ ) of functions. This is *not* a function in our typical sense; it ‘eats’ functions.... Functions ‘eat’ numbers.
- $\frac{d}{dx}(y)$ 
  - This is the derivative with respect to  $x$  of  $y$ ! This is exactly the same as  $\frac{dy}{dx}$ .
- $\left.\frac{dy}{dx}\right|_{x=a}$ 
  - This is the derivative with respect to  $x$  of  $y$  evaluated at  $x = a$ . Simply take the derivative of  $y$  with respect to  $x$ , then plug in  $a$  to the resulting function. Notice if  $y = f(x)$ , then  $\left.\frac{dy}{dx}\right|_{x=a} = f'(x)\big|_{x=a} = f'(a)$ .
- $\int f(x) dx$ 
  - This is the (general) antiderivative of  $f(x)$  with respect to  $x$ . Remember this is a set of *functions*. These typically end with ‘+  $c$ .’ This is *all* functions whose derivative are the integrand  $f(x)$ . We typically say ‘If  $F(x)$  is *an* antiderivative of  $f(x)$ , then  $\int f(x) dx = F(x) + c$ ,’ where  $c$  is any constant and  $F'(x) = f(x)$ .
- $\int_a^b f(x) dx$ 
  - This is the integral from  $x = a$  to  $x = b$  of  $f(x)$  with respect to  $x$ . This is interpreted as the ‘net area’ under the curve  $f(x)$  (to the  $x$ -axis) from  $x = a$  to  $x = b$  (when  $a < b$ ). If the curve is completely above the  $x$ -axis, then  $\int_a^b f(x)dx > 0$ , and if the curve is completely under the  $x$ -axis, then  $\int_a^b f(x)dx < 0$ . Notice that this is a *number(!)* and inherently different from the antiderivative.

## On the Chain Rule

Typically, people remember the chain rule as  $\frac{d}{dx}(f(g(x))) = f'(g(x)) \cdot g'(x)$ . This is true and useful for computing derivative. We can also see this as:

- $\frac{df}{dx} = \frac{df}{du} \cdot \frac{du}{dx}$
- This is the Chain Rule. This is saying that if  $f$  is a function that depends on  $u$  and  $u$  is a function of  $x$ , then we can take the derivative of  $f$  with respect to  $x$  since  $f$  is a function of  $x$  by the transitive property ( $f$  is a function of  $u$  which is a function of  $x$ , so  $f$  is a function of  $x$ ).

The Chain Rule then tells us that we can find  $\frac{df}{dx}$  by taking the derivative of  $f$  as a function of  $u$ , and then multiplying that by the derivative of  $u$  with respect to  $x$ . To have this make sense, before we say these are equal, we need to change the result of ‘the derivative of  $f$  with respect  $u$ ’ to be in terms of  $x$ .

**Example 1.**  $\frac{d}{dx}(u^3 + 7u^{1/2})$  where  $u = \sin(2x)$ .

If we call  $u^3 + 7u = f(u)$ , then by the chain rule  $\frac{df}{dx} = \frac{df}{du} \cdot \frac{du}{dx}$ .

Notice that  $\frac{df}{du} = 3u^2 + \frac{7}{2}u^{-1/2}$ , and  $\frac{du}{dx} = 2 \cos(2x)$ .

Notice that  $\frac{df}{du}$  is a function of  $u$ , but our answer should be a function of  $x$ , so

$$\begin{aligned} \frac{df}{du} \cdot \frac{du}{dx} &= \left(3u^2 + \frac{7}{2}u^{-1/2}\right) \left(2 \cos(2x)\right) \\ &= \left(3(\sin(2x))^2 + \frac{7}{2}(\sin(2x))^{-1/2}\right) (2 \cos(2x)) = \frac{df}{dx} \end{aligned}$$

□

- While it may *seem* to make sense to say that  $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$  is true because ‘the  $du$ ’s cancel,’ this makes no mathematical sense. The actual reason can be found in your textbook. The notation is just a tool to remember.

## Fundamental Theorem of Calculus: How It Works With The Chain Rule

**Example 2.** For the function  $y = \int_0^{\tan(x)} t^2 \sin(t) dt$ , find  $\frac{dy}{dx}$ .

First notice that FTC part 1 says that  $\frac{d}{dx} \left( \int_a^x f(t) dt \right) = f(x)$ . Particularly, the bottom limit of integration must be a constant and the upper limit of integration must be exactly the variable we are taking a derivative with respect to. But in the question at hand it is not. Let's first make the upper limit of integration be a regular variable, say  $u$ , so  $u = \tan(x)$ .

Recall that the chain rule says  $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$ . Let's find the first part of the product  $\left(\frac{dy}{du}\right)$  by switching all  $\tan(x)$ 's to  $u$ 's.

$$\frac{dy}{du} = \frac{d}{du} \left( \int_0^u t^2 \sin(t) dt \right) = (u)^2 \sin(u) = \tan(x)^2 \sin(\tan(x)).$$

Then  $\frac{du}{dx} = \frac{d}{dx}(\tan(x)) = \sec^2(x)$ , so  $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \left( \tan^2(x) \sin(\tan(x)) \right) \left( \sec^2(x) \right)$  □

## U-substitution: How To Make Hard Integrals Easier

This is the theorem that makes it all work.

**Theorem the substitution rule.** If  $u = g(x)$  is a differentiable function whose range is the interval  $I$  and  $f$  is continuous on  $I$ , then

$$\int f(g(x)) \cdot g'(x) dx = \int f(u) du.$$

*So why does it work?* Because it is the chain rule *backwards*. The most important feature about this is the the left integral is harder than the right one; we can take hard integrals and make them easier!

To use this in practice, we need to find a  $u$  so that everything works out well. Notice that there is a ' $du$ ' term. By the chain rule,  $du = \frac{du}{dx} dx$ . Therefore we need to pick  $u$  so that  $du$  is a good choice as well. Particularly the  $g'(x)dx$  terms becomes  $du$ :

$$\int f(g(x)) \cdot g'(x) dx = \int f(u) du.$$

**Example 3.** What is  $\int x^2 \cos(2x^3) dx$ ?

This integral is hard, so let's try to make it easier. Since we don't know any antiderivatives of trig functions besides ones with a constant multiple of  $x$ , i.e.  $\cos(kx)$ , then let's make it that way: call  $u = 2x^3$ , so  $\sin(2x^3) = \sin(u)$

Notice the the substitution rule has a  $du$  term, so we need to figure out who that bad boy is. All we need is the chain rule:  $du = \frac{du}{dx} dx$ . Notice that  $\frac{du}{dx} = 6x^2$ . Therefore

$$du = 6x^2 dx \quad \text{so} \quad dx = \frac{1}{6x^2} du$$

By substitution, we can then see that

$$\begin{aligned} \int x^2 \cos(2x^3) dx &= \int x^2 \cos(u) \frac{1}{6x^2} du = \int \cancel{x^2} \cos(u) \frac{1}{6\cancel{x^2}} du \\ &= \int \frac{1}{6} \cos(u) du = \frac{1}{6} \sin(u) + c = \frac{1}{6} \sin(2x^3) + c \end{aligned}$$

□

- Notice that we needed to get rid of all  $x$ 's in our expression before we could take the integral. Keeping that in mind, we needed to keep the ' $x^2$ ' in the expression so that it would cancel properly when we substituted in for  $dx$ .
- Further, our objective is to take an integral: when we make this  $u$  substitution, we actually want to take the integral. Lastly after we take the integral, we want our answer to be in terms of  $x$ , so substitute back in the proper function of  $x$ 's for  $u$ 's.
- Sometimes you might have to try multiple  $u$ 's until you find one that works. Sometimes you may even have to a  $u$ -substitution inside of a  $u$ -substitution (though you'd probably use a different variable the second time).

### Definite Integrals

This is just an application of the Fundamental Theorem of Calculus part 2. Recall that FTC part 2 says:

$$\int_a^b f(x) dx = F(b) - F(a) \quad \text{where } F'(x) = f(x), \text{ i.e. } F(x) \text{ is an antiderivative of } f(x).$$

For a definite integral, we may have to use  $u$ -substitution to find the antiderivative of  $f(x)$ , and then evaluate as FTC part 2 says to.

U-substitution version of FTC part 2

If  $u = g(x)$ , then

$$\int_a^b f(g(x)) \cdot g'(x) dx = \int_{g(a)}^{g(b)} f(u) du$$

- Notice that the limits of integration change! That is because we are translating the entire integral into  $u$ 's, and the left integral is from  $x = a$  to  $x = b$ . Therefore since  $u = g(x)$ , then  $g(a)$  and  $g(b)$  are  $a$  and  $b$  in the terms of  $u$ .

We will do two different styles of definite integral that have the same result.

**Example 4 by changing bounds and using '='.**

Find  $\int_1^3 x^2 \cos(2x^3) dx$ .

Recall that in Example 3, we found that  $\int x^2 \cos(2x^3) dx = \frac{1}{6} \sin(2x^3) + c$  by using  $u = 2x^3$ . Notice  $g(1) = 2$  and  $g(3) = 54$ . Then:

$$\int_1^3 x^2 \cos(2x^3) dx = \int_{g(1)}^{g(3)} \frac{1}{6} \cos(u) du = \int_2^{54} \frac{1}{6} \cos(u) du = \frac{1}{6} \sin(54) - \frac{1}{6} \sin(2)$$

since  $\frac{1}{6} \sin(u)$  is an antiderivative of  $\frac{1}{6} \cos(u)$ . □

- Advantages: The main advantage to this process is that you can actually use equalities throughout your steps! You can work on the problem right away without any preparations. Besides the choice of  $u$  and calculation of  $du$ , this process is a string of equalities.
- Disadvantages: You need to change your limits of integration.

Now to the next method.

**Example 4 by waiting until the end.**

Find  $\int_1^3 x^2 \cos(2x^3) dx$ .

Recall that in Example 3, we found that  $\int x^2 \cos(2x^3) dx = \frac{1}{6} \sin(2x^3) + c$ . Therefore  $\frac{1}{6} \sin(2x^3)$  is an antiderivative of  $x^2 \cos(2x^3)$ . Then:

$$\int_1^3 x^2 \cos(2x^3) dx = \left( \frac{1}{6} \sin(2x^3) \right) \Big|_{x=3} - \left( \frac{1}{6} \sin(2x^3) \right) \Big|_{x=1} = \frac{1}{6} \sin(2(3)^3) - \frac{1}{6} \sin(2(1)^3)$$

which is exactly the same result as in the other process.  $\square$

- Advantages: The advantage to waiting until the end is that you don't have to worry about the limits of integration. Simply wait until the end and then plug your bounds into an antiderivative function in terms of  $x$ .
- Disadvantages: You have to do a lot of prep work; you need to actually find the general antiderivative to the integrand before you do anything. You also have to convert everything back into terms of  $x$ . Further you can't use as many '=' signs, since the work doesn't flow as well. Therefore it is really easy to write an incorrect '=' and possibly lose points!

In general, either process works. But be warned: just because you don't need to find the new bounds in terms of  $u$  for the 'wait until the end' method, you still need to know how your bounds are changed. This is a fundamental concept in substitution of variables: you need to keep track of how changing from  $x$  to  $u$  affects  $du$  and bounds of integration.