Written HW 10 - Friday, November 3

Name: __________________________

1. The water trough in the figure below is made to be the dimensions shown in the figure. Only the angle $\theta$ can be varied. The value of $a$ is nonnegative and the value of $b$ is positive. Note that $1' = 1$ foot. [Retrieved from Thomas’ Calculus. CSU Special Edition. p.223]

(a) We wish to maximize the volume of water that the trough can hold. Find a function whose output represents the volume of the trough. It can be in terms of both $a$ and $b$.

$$\text{Volume} = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$$

(b) Rewrite $b$ in terms of $a$ and then use this to write the volume of the trough just in terms of $a$.

$$V(a) = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$$

(c) What if we rewrite the volume of the trough in terms of the angle $\theta$?

$$V(\theta) = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$$

(d) What is the domain of $V(\theta)$ (i.e. for the context of this problem, what is the set of values allowed for $\theta$)?

**Domain:** __________________________

(e) What value of $\theta$ will maximize the trough’s volume? Use calculus (i.e. use first and/or second derivatives) to justify your result. [*Hint: You will want to use the Pythagorean Identity $\cos^2(\theta) = 1 - \sin^2(\theta)$]*

$$\theta = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$$

(f) How is the use of $\theta$ in this problem different from how we’ve rewritten functions to be in one variable in other optimization problems?
2. Suppose that \( f(x) \) denotes a function that is continuous for all real numbers. The statement below is true sometimes. Give an example of a function for which it holds true and an example of a function for which it does not hold true. Explain your reasoning. Provide your answers by filling in the table below:

\[
\int_{a}^{b} f(x) \, dx \text{ will always give the total area enclosed by the curve and the x-axis.}
\]

<table>
<thead>
<tr>
<th>Example of True</th>
<th>Example of False</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
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Why is the statement true for your example? Why is the statement false for your example?
3. Use $f(x) = 2x^3 + 2x^2 - 12x$ on $[-3, 2]$ to answer the following questions.

(a) Estimate $\int f(x) \, dx$ on $[-3, 2]$ using $n = 5$ subintervals of equal length and left endpoints.

Be sure to draw the 5 rectangles you used on the graph provided and show all of the terms you included in your computation.
(b) Estimate $\int f(x) \, dx$ on $[-3, 2]$ using $n = 5$ subintervals of equal length and right endpoints.

Be sure to draw the 5 rectangles you used on the graph provided and show all of the terms you included in your computation.
(c) Find an upper estimate for the value of \( \int_{-3}^{2} f(x) \, dx \) using \( n = 5 \) subintervals of equal length.

Be sure to draw the 5 rectangles you used on the graph provided and show all of the terms you included in your computation.
(d) Find a lower estimate for the value of $\int_{-3}^{2} f(x) \, dx$ using $n = 5$ subintervals of equal length. Be sure to draw the 5 rectangles you used on the graph provided and show all of the terms you included in your computation.

(e) What is different about the way you draw the rectangles when you are finding an upper estimate or lower estimate compared to how you draw the rectangles when using right or left endpoints to approximate $\int_{-3}^{2} f(x) \, dx$? Describe in 2-5 sentences.
\[ f(x) = 2x^3 + 2x^2 - 12x \]

(f) \[ \int_{-3}^{2} f(x) \, dx = \frac{125}{6}. \]

In terms of the graph of \( f \) on the interval \([-3, 2]\), describe what the number \( \frac{125}{6} \) means.

(g) \[ \int_{-3}^{0} f(x) \, dx - \int_{0}^{2} f(x) \, dx = \frac{63}{2} - \frac{32}{3} = \frac{253}{6}. \]

In terms of the graph of \( f \) on the interval \([-3, 2]\), describe what the number \( \frac{253}{6} \) means.
4. The Riemann definite integral \( \int_a^b f(x)dx \) is defined as a limit of Riemann sums. The definite integral can be \textit{approximated} by a finite sum:

\[
\int_a^b f(x)dx \approx f(c_1)\Delta x_1 + f(c_2)\Delta x_2 + \ldots + f(c_{n-1})\Delta x_{n-1} + f(c_n)\Delta x_n
\]

where \( \Delta x_1 = (x_1 - x_0), \Delta x_2 = (x_2 - x_1), \ldots, \Delta x_{n-1} = (x_{n-1} - x_{n-2}), \Delta x_n = (x_n - x_{n-1}) \).

A function \( y = f(x) \) defined on the interval \([a, b] = [-1, 2]\) is shown in the figure.

(a) Explain in words how to interpret \( x_0, x_1, x_2, \ldots, x_{n-1}, x_n \). (Label these on the \( x \)-axis in the figure above with \( n = 6 \).)

(b) Explain in words how to interpret \( f(c_1), f(c_2), \ldots, f(c_{n-1}), f(c_n) \).

(c) In the figure above, draw in the rectangles that correspond to the finite sum when \( n = 6 \) and then label the widths of the bases and heights of your rectangles using the notation provided in the problem statement.

(d) The definite integral is defined as a limit of Riemann sums. What does the limit mean, and what purpose does it serve?

(e) Is \( \int_{-1}^{2} f(x)dx \) positive or negative? (Explain why in terms of the function \( f(x) \) and the \( x \)-axis.)