1. The water trough in the figure below is made to be the dimensions shown in the figure. Only the angle $\theta$ can be varied. The value of $a$ is nonnegative and the value of $b$ is positive. Note that $1' = 1$ foot. [Retrieved from Thomas’ Calculus, USM Special Edition, p. 223]

(a) We wish to maximize the volume of water that the trough can hold. Find a function whose output represents the volume of the trough. It can be in terms of both $a$ and $b$.

$$\text{Volume} = 20 \cdot 2 \cdot \frac{1}{2} ab + b : 1 = 20 (ab + b)$$

(b) Rewrite $b$ in terms of $a$ and then use this to write the volume of the trough just in terms of $a$.

$$b = \sqrt{1 - a^2} \quad \text{if } b > 0$$

$$V(a) = 20(1 - a^2 + \theta)$$

(c) What if we rewrite the volume of the trough in terms of the angle $\theta$?

$$\sin \theta = \frac{a}{1} \quad \cos \theta = \frac{b}{1}$$

$$V(\theta) = 20 \left( \frac{1 - \sin^2 \theta + \sin \theta (1 - \sin^2 \theta)}{1} \right)$$

(d) What is the domain of $V(\theta)$ (i.e. for the context of this problem, what is the set of values allowed for $\theta$)?

Domain: $0 \leq \theta \leq \frac{\pi}{2}$ and $0 \leq \theta < \frac{\pi}{2}$

(f) How is the use of $\theta$ in this problem different from how we’ve rewritten functions to be in one variable in other optimization problems?

Rather than writing $a$ in terms of $b$ (or vice versa), we were able to incorporate $\theta$ and thus write both $a$ & $b$ in terms of $\theta$.\n
2. Suppose that \( f(x) \) denotes a function that is continuous for all real numbers. The statement below is true sometimes. Give an example of a function for which it holds true and an example of a function for which it does not hold true. Explain your reasoning. Provide your answers by filling in the table below:

\[
\int_{a}^{b} f(x) \, dx \text{ will always give the total area enclosed by the curve and the } x\text{-axis.}
\]

<table>
<thead>
<tr>
<th>Example of True</th>
<th>Example of False</th>
</tr>
</thead>
<tbody>
<tr>
<td>![Diagram of a function with area under curve]</td>
<td>![Diagram of a line function]</td>
</tr>
</tbody>
</table>

Why is the statement true for your example?

If \( f(x) \) is completely on or above the \( x \)-axis, then \( \int_{a}^{b} f(x) \, dx \) will give a total enclosed area.

Why is the statement false for your example?

If \( f(x) \) is above and below the \( x \)-axis on \([a, b]\), then \( \int_{a}^{b} f(x) \, dx \) will give the difference between the area below and the area above.
3. Use \( f(x) = 2x^3 + 2x^2 - 12x \) on \([-3, 2]\) to answer the following questions.

(a) Estimate \( \int_{-3}^{2} f(x) \, dx \) on \([-3, 2]\) using \( n = 5 \) subintervals of equal length and left endpoints.

Be sure to draw the 5 rectangles you used on the graph provided and show all of the terms you included in your computation.

\[
\frac{2 - (-3)}{5} = \frac{5}{5} = 1
\]

\[
\int_{-3}^{2} f(x) \, dx \approx 1 \left( f(-3) + f(-2) + f(-1) + f(0) + f(1) \right)
\]

\[
\approx 1 \left( 0 + 12 + 0 + (-8) \right)
\]

\[
\approx 20
\]

(d) Find a lower estimate for the value of \( \int_{-3}^{2} f(x) \, dx \) using \( n = 5 \) subintervals of equal length. Be sure to draw the 5 rectangles you used on the graph provided and show all of the terms you included in your computation.

\[
\int_{-3}^{2} f(x) \, dx \approx 1 \left( f(-3) + f(-1) + f(0) + f(1) + f(1.20) \right)
\]

\[
\approx 1 \left( 0 + 12 + 0 + (-8) + (-8.12) \right)
\]

\[
\approx -9.121
\]
(g) \[ \int_{-3}^{0} f(x) \, dx - \int_{0}^{2} f(x) \, dx = \frac{63}{2} - \frac{32}{3} = \frac{253}{6}. \]

In terms of the graph of \( f \) on the interval \([-3, 2]\), describe what the number \( \frac{253}{6} \) means.

\[ \int_{-3}^{0} f(x) \, dx \] gives the area for region A.

\[ -\int_{0}^{2} f(x) \, dx \] translates region B into a positive value so we can interpret this as area.

\( \frac{253}{6} \) represents the total area between \( f(x) \) and the \( x \)-axis.
4. The Riemann definite integral \( \int_a^b f(x)\,dx \) is defined as a limit of Riemann sums. The definite integral can be approximated by a finite sum:

\[
\int_a^b f(x)\,dx \approx f(c_1)\Delta x_1 + f(c_2)\Delta x_2 + \ldots + f(c_{n-1})\Delta x_{n-1} + f(c_n)\Delta x_n
\]

where \( \Delta x_1 = (x_1 - x_0), \Delta x_2 = (x_2 - x_1), \ldots, \Delta x_{n-1} = (x_{n-1} - x_{n-2}), \Delta x_n = (x_n - x_{n-1}) \).

A function \( y = f(x) \) defined on the interval \([a, b] = [-1, 2]\) is shown in the figure.

(a) Explain in words how to interpret \( x_0, x_1, x_2, \ldots, x_{n-1}, x_n \). (Label these on the \( x \)-axis in the figure above with \( n = 6 \).)

Each \( x_i \) is an endpoint of a subinterval.

(b) Explain in words how to interpret \( f(c_1), f(c_2), \ldots, f(c_{n-1}), f(c_n) \).

\( f(c_i) \) is the height of the \( i \)th rectangle.

(c) In the figure above, draw in the rectangles that correspond to the finite sum when \( n = 6 \) and then label the widths of the bases and heights of your rectangles using the notation provided in the problem statement. <answers will vary>

(d) The definite integral is defined as a limit of Riemann sums. What does the limit mean, and what purpose does it serve?

The limit decreases the widths of the rectangles (widths \( \to 0 \)) & thus increases the \# of rectangles under the curve (\( n \to \infty \)). The more rectangles of smaller widths, the closer we get to the actual area under the curve. Something about error decreasing as the number of rectangles/pieces increases. Or talking about a finite approximation will always have error, so a limit is needed.

But have to be careful of this argument

(e) Is \( \int_{-1}^2 f(x)\,dx \) positive or negative? (Explain why in terms of the function \( f(x) \) and the \( x \)-axis.)

There is more area above the \( x \)-axis, so \( \int_{-1}^2 f(x)\,dx > 0 \).