Chain Rule & Implicit Differentiation

Let’s work on organizing our thoughts and computations to that, come Thursday, we may all stand victorious on a burning heap of derivatives.

1. Find \( \frac{df}{dx} \) for \( f(x) = \frac{3 \tan(2x^2+x)(-x^3-x^2)}{12x^5-13} \)

First, we rewrite our function in terms of “simpler” functions:

\[
\begin{align*}
\alpha(x) &= 3 \tan(x) \\
\beta(x) &= (2x^2 + x) \\
\gamma(x) &= (-x^3 - x^2) \\
\delta(x) &= (12x^5 - 13) \\
H(x) &= \alpha(\beta(x)) \gamma(x) \\
L(x) &= \delta(x)
\end{align*}
\]

which allows us to write

\[
f'(x) = \frac{L(x)H'(x) - H(x)L'(x)}{(L(x))^2} = \frac{[\delta(x)][\alpha'(\beta(x))\beta'(x)\gamma(x) + \gamma'(x) - \alpha(\beta(x))\gamma'(x)] - [\alpha(\beta(x))\gamma(x)][\delta'(x)]}{(\delta(x))^2}
\]
... and, so we see that we should probably find $\alpha' (\beta(x))$, $\beta' (x)$, $\gamma' (x)$, and $\delta' (x)$.

\[
\alpha' (x) = \frac{d}{dx} 3 \tan(x) = 3 \sec^2(x)
\]

\[
\Rightarrow \alpha' (\beta(x)) = 3 \sec^2(\beta(x)) = 3 \sec^2(2x^2 + x)
\]

\[
\beta' (x) = \frac{d}{dx} (2x^2 + x) = (4x + 1)
\]

(Notice that I kept parentheses around everything? This is so that I do not get confused when I plug $\beta'(x)$ back into my equation for $f'(x)$.)

\[
\gamma' (x) = \frac{d}{dx} (-x^3 - x^2) = (-3x^2 - 2x)
\]

\[
\delta' (x) = \frac{d}{dx} (12x^5 - 13) = (60x^4)
\]

Now, we can simply (or rather, laboriously) plug these into our equation for $f'(x)$ which we have in terms of our simpler functions. Remember to keep all parentheses and brackets, so that multiplication doesn’t go awry.

\[
f'(x) = \frac{\frac{d}{dx} H(x) - H(x) \frac{d}{dx} L(x)}{(L(x))^2}
\]

\[
= \frac{[\frac{d}{dx} (\alpha(\beta(x)) \beta(x) \gamma(x) + \gamma(x) \alpha(\beta(x))] - \alpha(\beta(x)) \gamma(x)]'}{(\beta(x))^2}
\]

\[
= \frac{12x^5 - 13 \{3 \sec^2(2x^2 + x)(4x + 1)(-3x^2 - 2x)\} + 3 \tan(2x^2 + x) - 3 \tan(2x^2 + x)(-3x^2 - 2x)\} [60x^4]
\]

\[
= \frac{12x^5 - 13}{(12x^5 - 13)^2}
\]

OK, so you might not get something this messy on the exam; but, if you can organize this, you can do anything the test might throw at you.

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2. Find $\frac{dy}{dx}$ for $(3x^2 + 7)^2 = 6y$

So, this looks like an implicit differentiation problem. In fact, it is.

\[
(3x^2 + 7)^2 = 6y
\]

(1) \[
\Rightarrow \frac{d}{dx}[(3x^2 + 7)^2] = \frac{d}{dx}[6y]
\]

(2) \[
\Rightarrow 2(3x^2 + 7) \frac{d}{dx}[3x^2 + 7] = 6 \frac{dy}{dx}
\]

(3) \[
\Rightarrow 2(3x^2 + 7)[3y + \frac{dy}{dx} 3x] = 6 \frac{dy}{dx}
\]

(4) \[
\Rightarrow 18x^2 y + \frac{dy}{dx} 18x^2 y + 42y + \frac{dy}{dx} 42x = \frac{dy}{dx} 6
\]

(5) \[
\Rightarrow \frac{dy}{dx} (18x^2 y + 42x - 6) = -18x^2 y - 42y
\]

(6) \[
\Rightarrow \frac{dy}{dx} = \frac{-18x^2 y - 42y}{18x^2 y + 42x - 6}
\]

In case you missed it, here’s a step-by-step breakdown of what we did:

(1) Differentiated each side with respect to $x$, treating $y$ as a function of $x$. 
(2) Used the chain rule on \( \frac{d}{dx}[(3x + y + 7)^2] \), where \( 2(3x + y + 7) \) is the derivative of the “outside,” and \( \frac{d}{dx}[3x + y + 7] \) is the derivative of the “inside.”

(3) Used the product rule and implicit differentiation to find \( \frac{d}{dx}[3x + y] \). (We used \( f(x) = 3x \) as our “first” function of \( x \) and \( g(x) = y \) as our “second” function of \( x \).)

(4) We foiled that junk.

(5) We put all terms with \( \frac{dy}{dx} \) (that for which we are solving) on one side of the equation so that we could factor out the \( \frac{dy}{dx} \)

(6) We divided both sides by \( (18x^2 + 42x - 6) \) in order to isolate \( \frac{dy}{dx} \).

Can you find \( y'' = \frac{d^2y}{dx^2} \)?