Show that Thm. 2.36 and its cor. are false if the word compact is replaced by closed or by dense.

bounded \( \cap n \in \mathbb{N} K_n = (0, 1/n) \) where \( n_0 \) is the largest \( n \) included and \( \cap n=1 K_n = \phi \).

closed \( \cap n \in \mathbb{N} K_n = \{ n, n+1, \ldots \} \) for \( n \in \mathbb{N} \). Then \( \cap \text{finite} K_n = \{ n_0, n_0+1, \ldots \} \) where \( n_0 \) is the largest \( n \) included and \( \cap n=1 K_n = \phi \).

\# 16. page 44. Regard \( \mathbb{Q} \) as a metric space with \( d(p, q) = |p - q| \). Let \( E = \{ p \in \mathbb{Q} \colon 2 < p^2 < 3 \} \). Show that \( E \) is closed and bounded in \( \mathbb{Q} \), but not compact. Is \( E \) open in \( \mathbb{Q} \)?

We consider \( \mathbb{Q} \) and \( d \) as a subspace of \( \mathbb{R} \). As a subset of \( \mathbb{R} \), \( E = \{ -\sqrt{3}, \sqrt{3} \} \cup (\sqrt{2}, \sqrt{3}) \) \( \cap \mathbb{Q} \).

Is \( E \) closed in \( \mathbb{Q} \). Choose \( p = 1/2 \). Then for \( x \in E \), \( d(x, p) \leq M = 17 \).

Consider \( E = \{ x \in \mathbb{Q} \colon x^2 > 3 \} \cup \{ x^2 < 2 \} \). Let \( G \subset \mathbb{R} \)
be defined as \( G = (-\infty, -\sqrt{3}) \cup (\sqrt{2}, \sqrt{3}) \cup (\sqrt{3}, \infty) \). \( G \) is open in \( \mathbb{R} \). As a subset of \( \mathbb{R} \), \( E = G \cap \mathbb{Q} \). Thus by Thm 2.30 (Prop 21) \( E \) is open relatively to \( \mathbb{Q} \).

Thus \( E \) is closed in \( \mathbb{Q} \).

Note that \( E \subset \mathbb{R} \cap \mathbb{Q} \). \( E \) is not compact in \( \mathbb{R} \) (\( E \) is not closed in \( \mathbb{R} \) because it's like \( \mathbb{Q} \)). Then by Thm 2.33 (Prop 22) \( E \) is not compact in \( \mathbb{Q} \).

\( E \) is open in \( \mathbb{Q} \) by the same approach used to prove that \( E \subset \mathbb{R} \) open in \( \mathbb{Q} \).