#1. \( \pm 1, \) pg. 165. \{f_n\} converges uniformly on \( E \subseteq \mathbb{R} \), \( f_n \) hold, \( n=1, \ldots \)

Pt. 1. \( \exists f_3 \) \( \forall n \geq N \). Let \( \epsilon = 1 \). \( \exists N \) s.t. \( n, m \geq N \Rightarrow |f_n(x) - f_m(x)| < \epsilon = 1 \).

Then for \( n \geq N \)

\[
|f_n(x) - f_N(x)| \leq |f_n(x) - f_N(x)| < 1.
\]

or \( |f_n(x)| = 1 + f_N(x) \) for all \( n \geq N \) and \( x \in E \).

Pt. 2. \( \exists f_4 \) \( \forall n \geq N \). \( \exists M_N \) s.t. \( |f_n(x)| \leq M_N \) for \( x \in E \).

Pt. 3. \( \exists f_5 \) \( \forall n \geq N \). \( \exists M_{N-1} \) s.t. \( |f_{N-1}(x)| \leq M_{N-1} \) for \( x \in E \), \( j=1, \ldots, N \).

Let \( M = \max \{M_1, \ldots, M_{N-1}, M_N \} + 1 \).

Then \( |f_n(x)| \leq M \) for all \( x \in E \), \( n=1, 2, \ldots \).

#3. \( \exists f_3, \) \( \exists f_4 \) converge uniformly on \( E \). Show that it is not the case that \( \{f_n\} \) converges uniformly on \( E \).

Note: By #2 (if it is correct), one of the sequence should not be held.

1. Choose \( f_n(x) = x \), \( n=1, 2, \ldots \) Clearly \( f_n \) uniformly.

Claim: \( f(x) = \lim f_n(x) \) where \( f(x) = x \).

Claim: \( g_n(x) = \frac{x}{n} \), \( n=1, 2, \ldots \), Clearly \( f_n \) \( g_n \) uniform on \( R \).

Claim: \( \{f_n\} \) does not converge uniformly.

Let \( \epsilon = 1 \) and consider any \( N \). Then consider \( n = N, m = 2N \) and \( x = 4N \). Then \( m, n \geq N \) and

\[
|\frac{x}{n} - \frac{x}{m}| = \left| \frac{(m-n)x}{nm} \right| = \left| \frac{N \cdot 4N}{N \cdot 2N} \right| = 2 > \epsilon = 1.
\]

Thus \( \{f_n\} \) is not a unique C.S. Thus \( \{f_n\} \) does not converge uniformly.
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\[ f(x) = \sum_{n=1}^{\infty} \frac{1}{1+n^2x} \]

(a) For which values of \( x \) does the series converge absolutely.

Clearly, \( \sum \frac{1}{1+n^2x} \) does not converge at \( x = 0 \), \( x = -\frac{1}{n^2} \), \( n \in \mathbb{N} \).

Otherwise, we have

\[ \left| \frac{1}{1+n^2x} \right| < \frac{1}{1\cdot n^2} \]

\[ \sum \frac{1}{1+n^2x} \text{ conv.} \Rightarrow \sum_{n=1}^{\infty} \frac{1}{1+n^2x} \text{ conv. abs. by the comparison test on } \mathbb{R} \text{ for } x \neq 0 \text{ and } x = -\frac{1}{n^2}, n \in \mathbb{N} \]

(b) On which intervals does it converge uniformly?

(i) \([-\infty, -1) \cup [1, \infty)\) for any \( a > 0 \):

\[ \left| \frac{1}{1+n^2x} \right| = \frac{1}{1\cdot n^2} < \frac{1}{n^2} \quad \text{if } n \geq 1. \]

\[ \sum_{n=2}^{\infty} \frac{1}{n^2} \text{ conv. } \Rightarrow \sum_{n=2}^{\infty} \frac{1}{1+n^2x} \text{ conv. uniformly by Prop. 89.} \]

\[ \sum_{n=1}^{\infty} \frac{1}{1+n^2x} \quad \text{one extra term wouldn't hurt it.} \]

(ii) \((\frac{-1}{m^2}, \frac{1}{(m+1)^2})\) for \( m \in \mathbb{N} \) \( m + 1 > n_0 \):

\[ x \in \left( -\frac{1}{m^2}, -\frac{1}{(m+1)^2} \right) \Rightarrow \left| 1x - \frac{1}{(m+1)^2} \right| > \frac{1}{n_0^2} \quad \text{and} \quad \frac{1}{1\cdot x} < n_0^2. \]

If \( n > n_0 \), then \( n^2 > n_0^2 \Rightarrow \left(\frac{1}{(m+1)^2}\right)^2 > \frac{1}{X} \), or \( -\frac{1}{X} < -\frac{1}{(m+1)^2} \) - This implies that \( 0 < n^2 - n_0^2 = n^2 + \frac{1}{X} \) or \( \frac{1}{n^2 + \frac{1}{X}} < \frac{1}{n^2 - n_0^2} \). This also shows that

\[ \left| \frac{1}{n^2 + \frac{1}{X}} \right| = \frac{1}{n^2 + \frac{1}{X}} < n_0^2 \quad \text{for } \frac{1}{n^2 + \frac{1}{X}} < \frac{n_0^2}{n^2 - n_0^2}. \]

Then

\[ \left| \frac{1}{1+n^2x} \right| = \frac{1}{1\cdot x} \left| \frac{1}{n^2 + \frac{1}{X}} \right| < n_0^2 \quad \text{for } \frac{1}{n^2 + \frac{1}{X}} < \frac{n_0^2}{n^2 - n_0^2}. \]
Then since \( \sum_{n=1}^{\infty} \frac{1}{n^2 + n^3} \) converges by Prop 8.9, we know that \( \sum_{n=1}^{\infty} \frac{1}{n^2 + n^3} \) converges uniformly on \((-\frac{1}{m^2}, -\frac{1}{(m+1)^2})\).

And again, no terms can't affect uniform convergence, so \( \sum_{n=1}^{\infty} \frac{1}{1+n^2} \) converges uniformly on \((-\frac{1}{m^2}, -\frac{1}{(m+1)^2})\).

Note: The interval in (i) cannot be extended to \((0, \infty)\). In fact, the series does not converge uniformly on any interval \((0, a] \) for any \( a > 0 \).

Let \( a = \frac{1}{2} \) and consider any \( N \in \mathbb{N} \). Then for \( m, n > N \), say \( m > n \), we can choose \( x = \frac{1}{2n^2} \) and see that

\[
\left| \sum_{j=n}^{m} \frac{1}{1+j^2} \right| = \sum_{j=n}^{m} \frac{1}{1+j^2} \leq \sum_{j=n}^{m} \frac{1}{1+\frac{1}{2n^2}m^2} \leq \sum_{j=n}^{m} \frac{1}{1+\frac{1}{2n^2}j^2} \leq \frac{1}{n-n^2} \cdot \frac{2}{3} > \frac{1}{2}.
\]

So \( \sum_{n=1}^{\infty} \frac{1}{1+n^2} \) is not uniformly convergent on \((0, a] \). Therefore, \( \sum_{n=1}^{\infty} \frac{1}{1+n^2} \) converges uniformly on \((a, \infty)\) for any \( a > 0 \), \((a, 0)\), \((-\infty, 0)\), and \((-\frac{1}{m^2}, -\frac{1}{(m+1)^2})\), \(m = 1, 2, \ldots\).

Of course there are "stupid" intervals on which it fails to converge uniformly, such as \((-1, -\frac{1}{2})\), etc—but we won't list these.

(c) If \( f \) is not convergent when ever the conv. is uniform.

(d) \( f \) is not bounded on \((a, \infty)\) but is not bounded on the other intervals.

> Since the conv. is uniform on \((a, \infty)\) for any \( a \), \( f \) is not convergent on \((0, \infty)\).