Problem 1.3.1 (§2.1, p. 22) Prove that there is no rational number \( \sqrt{2} \) s.t. \( x^2 = 12 \).

Proof: Suppose \( x \) is rational, \( x = \frac{m}{n} \) in reduced form. Then \( m^2 = 12n^2 \). Since \( m^2 \) is even, \( m \) is even. Let \( m = 2k \). Then \( 4k^2 = 12n^2 \) or \( k^2 = 3n^2 \).

If \( k^2 \) is divisible by 3, \( k \) is divisible by 3. (Write \( k^2 \) in terms of its prime factors — which will be the prime factors of \( k \). One of the prime factors must be a 3. Then there is a 3 in the factorization of each \( k \).) Then \( k = 3p \) or \( 9p^2 = 3n^2 \) or \( n^2 = 3p^2 \).

Since \( n^2 \) is divisible by 3, \( n \) is divisible by 3. Let \( n = 3q \).

Here we have \( n = 3q \) and \( m = 2k = 2 \cdot 3 \cdot p \). This contradicts the fact that \( m/n \) is in reduced form.

Try again: Consider the polynomial eqn \( x^2 - 12 = 0 \).

Then by Prop 1.1.1 in the notes if \( r \in \mathbb{Q} \) is a root of the eqn \( x^2 - 12 = 0 \) where \( r = \frac{b}{a} \) is in reduced form, then \( g \) divides \( a_0 = 1 \) and \( b \) divides \( a_n = 12 \), i.e. the rational roots of \( x^2 - 12 = 0 \) must come from the set \( \pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 12 \). Trying all \( g \) there shows that \( x^2 - 12 = 0 \) has no rational roots, i.e. no rational \( x \) s.t. \( x^2 = 12 \).
2. (#4, p.22) Let \( E \neq \emptyset \) be a subset of an ordered set. Suppose \( \alpha \) is a lower bd of \( E \) and \( \beta \) is an upper bd. of \( E \). Then \( \alpha \leq \beta \).

Pf: Assume false. Assume \( \alpha > \beta \). Since \( \alpha \) is a lower bd of \( E \), \( x \in E \Rightarrow \alpha \leq x \). Then \( \alpha > x > \beta \). This contradicts the fact that \( \beta \) is an upper bd of \( E \).

#3. Suppose \( E \neq \emptyset \) is a subset of a complete ordered field. Prove that \( m^* = \text{glb}(E) \leq \text{lub}(E) = M^* \).

Pf: Suppose false. Suppose \( m^* > M^* \). Since \( m^* = \text{glb}(E) \) for every \( \varepsilon > 0 \) there exists \( x \in E \) s.t. \( \varepsilon - m^* < \varepsilon \). Then \( x > m^* > M^* \). This contradicts the fact that \( M^* = \text{lub}(E) \) — \( M^* \) must be an upper bd of \( E \).

#3. Done right. This \( ^* \) is correct but is not the simplest way to do it. So we'll try again.

E is bdd \( \Rightarrow m^* \) and \( M^* \) exist. We know then that \( m^* \) is a lower bd of \( E \) and \( M^* \) is an upper bd of \( E \). By proble \#2, \( m^* \leq M^* \).