#12. Suppose \( X, Y, Z \) are metric spaces, \( E \subset X \), 
\( f ; E \rightarrow Y, g ; f(E) \rightarrow Z \) and \( h = g \circ f ; E \rightarrow Z \).
Prove that if \( f \) is unif. cont. on \( E \) and \( g \) is unif. cont. on \( f(E) \), then \( h \) is unif. cont. on \( E \).

**Proof:** Suppose \( \varepsilon > 0 \) is given. \( g \) unif. cont. on \( f(E) \) \( \Rightarrow \exists \delta_1, s. t., d(y_1, y_2) < \delta_1 \Rightarrow d(g(y_1), g(y_2)) < \varepsilon \).

\( f \) unif. cont. on \( E \) (applied with the traditional "\( \varepsilon \)"
replaced by \( \delta_1 \)) gives us a \( \delta \) s. t. \( d_x(x_1, x_2) < \delta \Rightarrow d_y(f(x_1), f(x_2)) < \delta_1 \).

Then if \( x_1, x_2 \in E \) s. t. \( d_x(x_1, x_2) < \delta \), \( f(x_1) \) and \( f(x_2) \) satisfy \( \delta \) above (with \( f(x_1), f(x_2) \) replacing \( y_1 \) and \( y_2 \), respectively) so \( d_y(g(f(x_1)), g(f(x_2))) < \varepsilon \).

Thus \( g \circ f \) is unif. cont. on \( E \).

#1.

\[
\lim_{x \to 3} \frac{x^3 - 3}{x^3 - 2x^2 - 2x - 3} = \lim_{x \to 3} \frac{x^3}{(x-3)(x^2 + x + 1)} = \frac{1}{13}
\]

Suppose \( \varepsilon > 0 \) is given. We note that

\[
\left| \frac{x^3}{(x-3)(x^2 + x + 1)} - \frac{1}{13} \right| = \left| \frac{13x^3 - (x^2 + x + 1)}{13(x-3)(x^2 + x + 1)} \right|
\]

Choose \( s_1 = \frac{1}{13} \) and \( x-3 < s_1 \) or \( x \in (2, 4) \), to see that the

\[
\max_{x \in [1, 4]} |x + 4| = 8
\]

and the min of \( x^2 + x + 1 \) is 7.

Thus \( \frac{x^3}{x^3 - 2x^2 - 2x - 3} - \frac{1}{13} \leq \varepsilon \) for all \( x \) if \( x-3 < \delta \), where \( \delta \) is chosen by the last inequality.

Thus \( \lim_{x \to 3} \frac{x^3}{x^3 - 2x^2 - 2x - 3} = \frac{1}{13} \).
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\[ f(x) = \begin{cases} 0 & \text{if } x \text{ irrational} \\ \frac{1}{n} & \text{if } x = \frac{m}{n}, \text{ reduced form } \frac{0}{1} \end{cases} \]

Prove that \( f \) is count at irrational pts and discontinuous at \( \frac{1}{n} \) of first kind at rational pts.

Proof: Let \( b \in \mathbb{R} \) and consider \( I = (b-1, b+1) \). Suppose \( \epsilon > 0 \) is given. By Archimedes there exists \( n_0 \in \mathbb{N} \) such that \( \frac{1}{n_0} < \epsilon \). There are only a finite number of \( n \)'s in \( \mathbb{N} \) such that \( \frac{1}{n} > \frac{1}{n_0} \) (the \( n \)'s less than \( n_0 \)).

In \( I \) there are only a finite number of rationals \( \frac{m}{n} \) where \( \frac{1}{n} > \frac{1}{n_0} \). The rationals \( \frac{m}{2} \) (since \( \text{the length of } I \text{ in } \mathbb{Z} \) must be a finite number), the rationals \( \frac{m}{3} \), etc., the rationals \( \frac{m}{n_0-1} \) (same description).

Since there are a finite number of rationals, we get \( s = \max \{ |b - \frac{m}{n}| : \frac{m}{n} \text{ reduced with } \frac{1}{n} > \frac{1}{n_0}, b + \frac{m}{n} \} \).

By construction if \( x \in \mathbb{Q}, |x-b| < \delta \) and \( x = \frac{m}{n} \), then \( \frac{1}{n} = \frac{1}{n_0} \) i.e., \( f(x) = \frac{1}{n} = \frac{1}{n_0} < \epsilon \).

Thus, for \( b \text{ in } \mathbb{R} \), \( x \text{ satisfies } |x-b| < \delta \), then \( |f(x)-f(b)| = |\frac{1}{n} - 0| = |x - b| < \epsilon \).

Thus, \( f \) is count at all rationals.

If \( b \) is not, if \( x \text{ satisfies } 0 < |x-b| < \delta \), then
\[ |f(x) - 0| = \left| \frac{1}{n} \right| < \epsilon \]
Thus, \( f(b+) = 0 \).
Likewise \( f(b^-) = 0 \). Since \( f(b) = \frac{1}{n} \) and \( b = \frac{\sqrt{2}}{n} \) which is \( 0 \), \( f \) is discontinuous at all rational \( b \). Since \( f(b^+) \) and \( f(b^-) \) exist, the discontinuities are 1st kind.

Note: You can also easily prove that \( f \) is discontinuous at \( b = b \) by considering the seq \( \{ b + \frac{\sqrt{2}}{n^2} \} \).