Prone: f is cont on E iff the graph of f is compact.
Denote the graph of f on E by G^f E.

Pf:

(⇒) Define h: E → X × Y by h(x) = (x, f(x)).
Define the metric d on X × Y by
\[ d((x_1, y_1), (x_2, y_2)) = d_X(x_1, x_2) + d_Y(y_1, y_2) \]
where d_X and d_Y are the metrics on X × Y, resp. (It's pretty easy to see that d will define a metric — you might check it out.)
Then
\[ d((x, f(x)), (x_0, f(x_0))) = d_X(x, x_0) + d_Y(f(x), f(x_0)). \]
Suppose ε > 0 is given. f cont on E ⇒ for x_0 ∈ E
\[ \exists \delta \ s.t. \ d_X(x, x_0) < \delta \ and \ x ∈ E ⇒ d_Y(f(x), f(x_0)) < \varepsilon/2. \]
Then if we choose \[ \delta = \min \{ \delta, \varepsilon/2 \} \] we see that
\[ d((x, f(x)), (x_0, f(x_0))) = d_X(x, x_0) + d_Y(f(x), f(x_0)) < \varepsilon/2 + \varepsilon/2 ≤ \varepsilon. \]
Hence h: E → X × Y is cont.
Then since E is compact, h(E) is compact. It is easy to see that h(E) = G^f E, the graph of f on E.

(⇐) Suppose f is cont continuous at x_0 ∈ E s.t. x_0 ∈ E'.
(f if x_0 is an isolated pt of E, then f is cont at x_0). Then there exists a ray \[ ε_0 > 0 \ s.t. \ \d_X(\alpha, x_0) > ε_0 \ for \ all \ \alpha \] and \[ f(\alpha) → f(x_0), \ i.e. \ there \ exists \ an \ ε_0 > 0 \ s.t. \ for \ every \ N ∈ \mathbb{N} \]
\[ \exists n_0 \ s.t. \ d((f(x_n)), f(x_0)) > ε_0. \]
Choose N = 1 9ε_1 n_1 s.t. d((f(x_n)), f(x_0)) > ε_0
Choose N = 2 9ε_2 n_2 s.t. d((f(x_n)), f(x_0)) > ε_0
eq ε_0 9ε_0 s.t. d((f(x_n)), f(x_0)) > ε_0
eq ε_0 9ε_0 s.t. d((f(x_n)), f(x_0)) > ε_0
etc. get subseq \[ x_{n_1}, x_{n_2}, \ldots \] s.t. d((f(x_n)), f(x_0)) > ε_0.
Now consider the seq $\{ (x_{n_j}, f(x_{n_j})) \}$ in $X \times Y$.

Since $G_{E}^f$ is compact, by Thm 3.6(a) there exists a subseq $\{ (x_{n_{k_j}}, f(x_{n_{k_j}})) \}$ that converges to a pt $\,(y, f(y))$ in $G_{E}^f$. Since, by def.,

$$d_X(x_{n_{k_j}}, y) \leq d_X((x_{n_{k_j}}, f(x_{n_{k_j}})), (y, f(y))) \leq d((x_{n_{k_j}}, f(x_{n_{k_j}})), (y, f(y)))$$

(where $d$ is the metric defined in the other direction), $x_{n_{k_j}} \to y$ and $f(x_{n_{k_j}}) \to f(y)$.

But $x_n \to x_0$ so it must be the case that $x_0 = y$. Thus $f(x_{n_{k_j}}) \to f(x_0)$. This is a contradiction to the fact that $d((f(x_{n_{k_j}}), f(x_0)) > 0$

since $\{x_{n_{k_j}}\}_{k=1}^\infty$ is a subseq of $\{x_{n_j}\}_{j=1}^\infty$.

Thus $f$ is cont on $E$. 
2.

If \( f: \mathbb{R} \to \mathbb{R} \), \( E \subseteq \mathbb{R} \)bdd. uniform cont. Then for \( E \) bdd. cont. \( f \) is continious on \( E \).

Thus, if \( E \) bdd. \( f \) cont. give \( \epsilon > 0 \) \( \exists \delta > 0 \) s.t. \( x, y \in E \), \( |x - y| < \delta \Rightarrow |f(x) - f(y)| < \epsilon \).

If \( E \)bdd. \( \exists M \) s.t. \( x \in E \Rightarrow -M < x < M \). Then \( \exists (x_1, \ldots, x_n) \) s.t. \( \bigcup_{i=1}^{n} (x_i - \delta, x_i + \delta) \supseteq [-M, M] \). For any \( i \in \{1, \ldots, n\} \), \( x \in (x_i - \delta, x_i + \delta) \Rightarrow |x - x_i| < \delta \). Then \( |f(x) - f(x_i)| \leq 1 \).

\( |f(x) - f(x_i)| \leq |f(x) - f(x_i)| + |f(x_i) - f(x_i)| = |f(x) - f(x_i)| + |f(x_i)| \leq 2|f(x)| + |f(x_i)| \leq 2|f(x)| \leq 2\epsilon \).

Let \( K = \max \{|f(x)| : x \in [-M, M] \} \). Then \( |f(x)| \leq K \).

If \( f: \mathbb{R} \to \mathbb{R} \), \( E \subseteq \mathbb{R} \) finite cont. Then \( f(E) \) is not necessarily bdd. \( f(x) = x \), \( E = \mathbb{R} \).
HW 17 # 3. Prove: \( f(x) = \frac{1}{1+x^2} \) is uniformly continuous on \( \mathbb{R} \).

\[
f(x) = f(y) = \frac{1}{1+x^2} - \frac{1}{1+y^2} = \frac{x^2-y^2}{(1+x^2)(1+y^2)} = (x-y) \frac{x+y}{(1+x^2)(1+y^2)}
\]

Claim: \( \left| \frac{x+y}{(1+x^2)(1+y^2)} \right| \leq 2 \)

\[
\frac{x+y}{(1+x^2)(1+y^2)} \leq 2
\]

5.9. \( x+y \geq 2x^2 + y^2 + 2x^2y^2 \) \( \geq 2 - 2x^2 - y^2 + 2x^2y^2 \)

\[
\frac{x+y}{(1+x^2)(1+y^2)} \geq 2 \quad \text{If } x+y \geq -2 \quad \text{5.9. } x+y \geq -2 - 2x^2 - y^2 + 2x^2y^2
\]

5.9. \( x+y + 2 + 2x^2 + y^2 + 2x^2y^2 \geq 0 \) \( \text{or } (x+y+\frac{1}{2})^2 + x^2 + y^2 + \frac{1}{4} - 2xy + 2x^2y^2 \)

\[
= (x+y+\frac{1}{2})^2 + (x-y)^2 + \frac{1}{4} + 2x^2y^2
\]

5.9. \( x+y + 2 + 2x^2 + y^2 + 2x^2y^2 \geq 0 \) \( \text{This is true} \quad \text{thus } \frac{x+y}{(1+x^2)(1+y^2)} \geq 2 \)

Thus \( \left| f(x) - f(y) \right| \leq 2|x-y| \). Let \( \varepsilon > 0 \) be given. Let \( \delta = \varepsilon/2 \).

Then \( |x-y| < \delta = \varepsilon/2 \Rightarrow \left| f(x) - f(y) \right| \leq 2|x-y| < \frac{\varepsilon}{2} = \varepsilon \).

Thus \( f(x) \) is uniformly continuous on \( \mathbb{R} \).