HW

#7. \( a_n \geq 0, n = 1, \ldots \) and \( \Sigma a_n \) converges \( \Rightarrow \) \( \sum \frac{\sqrt{a_n}}{n} \) converges.

Pf:

Solve 1: \( \Sigma a_n \) converges \( \Rightarrow \) \( S_n = \sum_{m=1}^{n} a_m \) is held. and \( S_n \rightarrow S \), for some \( S \).

\( \Sigma \frac{1}{m} \) converges \( \Rightarrow \) \( T_n = \sum_{m=1}^{n} \frac{1}{m} \) is held and \( T_n \rightarrow T \), for some \( T \).

By Schwarz's Ineq.

\[
\left| \sum_{m=1}^{n} \frac{1}{m} \sqrt{a_m} \right|^2 \leq \sum_{m=1}^{n} \frac{1}{m^2} \sum_{m=1}^{n} (\sqrt{a_m})^2
\]

or

\[
\sum_{m=1}^{n} \frac{1}{m} \sqrt{a_m} \leq \sqrt{T \sum \sqrt{a_m}} \quad \text{Using the fact that everything around is} \geq 0
\]

Then the seq \( Q_n = \sum_{m=1}^{n} \frac{1}{m} \sqrt{a_m} \) is held above and monotone incr. Thus \( \lim_{n \to \infty} Q_n \) exists, i.e. \( \sum_{m=1}^{\infty} \frac{1}{m} \sqrt{a_m} \) converges.

Solve 2: \( \Sigma a_n \) converges \( \Rightarrow \) for every \( \varepsilon > 0 \), \( \exists N_1, s.t. m, n \geq N_1 \Rightarrow \left| \sum_{j=m}^{n} a_j \right| < \varepsilon \)

Since \( \Sigma \frac{1}{j} \) converges.

Also \( \exists N_2 \), s.t. \( m, n \geq N_2, m \geq n \Rightarrow \left| \sum_{j=m}^{n} \frac{1}{j} \right| < \varepsilon \). (Both by Cauchy's Crit.)

Then by Schwarz's Ineq. for \( n, m \geq \max \{N_1, N_2\}, n \geq m,

\[
\left| \sum_{j=m}^{n} \frac{1}{j} \sqrt{a_j} \right|^2 \leq \sum_{j=m}^{n} \frac{1}{j^2} \sum_{j=m}^{n} a_j \leq \varepsilon \cdot \varepsilon = \varepsilon^2,
\]

or

\[
\left| \sum_{j=m}^{n} \frac{1}{j} \right| \leq \sqrt{\frac{\varepsilon^2}{n-m+1}} \leq \frac{\varepsilon}{\sqrt{n-m+1}} \quad \text{converg by the Cauchy criteria}.
\]
Let \( A = \frac{1}{2} \), then \( |S_n| \leq A \)

Then

\[
\left| \sum_{n=p}^{q} a_n B_n \right| \leq \left| \sum_{n=p}^{q} S_n (B_n - B_{n+1}) \right| + \sum_{n=p}^{q} B_n - \frac{1}{2} \sum_{n=p}^{q} B_{n+1}
\]

\[
\leq A \left[ \sum_{n=p}^{q-1} (B_n - B_{n+1}) \right] + B_p + B_{p-1}
\]

\[
= A \left[ B_p - \frac{B_p}{2} \right] + B_p + B_{p-1} = A \left( B_p + B_{p-1} \right)
\]

Let \( \epsilon > 0 \) be given.

\( B_n \to 0 \) \( \Rightarrow \) \( \exists \ N \ s.t. \ n \geq N \Rightarrow |B_n| < \frac{\epsilon}{2A} \)

Then for \( p, q \geq N, q \geq p \Rightarrow \left| \sum_{n=p}^{q} a_n B_n \right| \leq A \left( \frac{\epsilon}{2A} + \frac{\epsilon}{2A} \right) = \epsilon
\]

Thus by the Cauchy criterion \( \sum a_n b_n \) converges.

Since \( \sum a_n, \sum b_n, \sum a_n b_n \) converges \( \Rightarrow \sum_{n=1}^{\infty} (a_n b_n - b_n a_n) \) converges.

\[
\sum_{n=1}^{\infty} (a_n b_n - b_n a_n) = \sum_{n=1}^{\infty} a_n (b_n - a_n) \]

\[
\Rightarrow \sum_{n=1}^{\infty} a_n b_n \text{ converges}
\]
If \( b_n \) is monotonically decreasing, let \( B_n = b_n - b_1 \).

Same proof: we find that \( \sum_{n=1}^{\infty} a_n B_n \) is convergent. Then because \( \sum_{n=1}^{\infty} b_n \) is convergent,
\[
\frac{\sum_{n=1}^{\infty} (a_n b_n + b_n a_n)}{n=1} = \sum_{n=1}^{\infty} a_n b_n \text{ converges.}
\]

To prove convergent by applying Theorem 3.42, define \( B_n \) as above (both parts), we get
\[
\text{conv} \sum_{n=1}^{\infty} a_n B_n \text{ as we did above. Then again we must proceed as we did above to get that }
\]
\[
\sum_{n=1}^{\infty} a_n b_n \text{ converges — in both cases,}
\]
i.e. Then 3.42 lets you skip from \( \sum \) to \( \sum \) — for the rest, see. The rest must still be done.