HW 7: Let \( S \neq \emptyset \), \( S \) closed (above and below). Prove that \( \inf S = \sup S \).

**Proof:** \( S \neq \emptyset \Rightarrow \exists x \in S \). Let \( g^* = \inf S \) and \( e^* = \sup S \).

\( g^* \) is a lower bd. of \( S \Rightarrow g^* \leq x \). \( e^* \) is an upper bd. of \( S \Rightarrow x \leq e^* \). \( \Rightarrow g^* \leq x \leq e^* \) \( \Rightarrow g^* \leq e^* \).

HW 8: Prove that \( \sqrt{2} \) is irrational.

Recall the theorem: Suppose \( a_0, a_1, \ldots, a_n \) are integers and \( r = \frac{p}{q} \), in reduced form, is a rational number that satisfies

\[ a_n x^n + a_{n-1} x^{n-1} + \ldots + a_0 = 0. \]

Then \( q \) divides \( a_n \) and \( p \) divides \( a_0 \).

If \( x = \sqrt{2} \), then \( x \) must satisfy the eqn \( x^2 - 2 = 0 \).

Apply the above result. Look for a rational root \( r = \frac{p}{q} \).

Then \( q \) divides \( a_n = 1 \) and \( p \) divides \( a_0 = 2 \).

i.e., we have \( r = \pm 1, \pm 2 \); \( (-1)^3 - 2 = -3 \neq 0 \), \( (2)^3 - 2 = -10 \neq 0 \), \( (-2)^3 - 2 = -6 \neq 0 \), \( (1)^3 - 2 = -1 \neq 0 \), \( (2)^3 - 2 = 6 \neq 0 \).

There are no rational roots to the eqn \( x^3 - 2 = 0 \).

\( \sqrt{2} \) (which clearly is a root) is not rational.