HW 62. \( f : [a, b] \to \mathbb{R} \), f cont on \([a, b]\) and \( f(x) \geq k \) for all \( x \in [a, b] \),

Then \( \int_{a}^{b} f \geq k(b-a) \),

\( \text{Pf:} \) Define \( g(x) = k \) for \( x \in [a, b] \), Then \( \int f \geq \int g \) and \( g \) is cont.
Then by Prop. 6.18: \( \int_{a}^{b} f \geq \int_{a}^{b} g = \int_{a}^{b} k \, dx = k(b-a) \).

b) \( f : [a, b] \to \mathbb{R} \), cont on \([a, b]\) and \( \int_{a}^{b} f = 0 \). Then \( \exists x_0 \in [a, b] \)

\( f(x_0) = 0 \).

f cont on \([a, b] \Rightarrow \) by the Extreme Value Thm. Then \( f \geq 0 \) there exists \( x_0, x \in [a, b] \) s.t. \( f(x_0) = \sup \{ f(x) : x \in [a, b] \} \)

\( f(x) = \sup \{ f(x) : x \in [a, b] \} \). 

If \( f(x) > 0 \), then \( f(x) > f(x_0) \) for all \( x \in [a, b] \)

and by part (a) \( \int_{a}^{b} f \geq \int_{a}^{b} f(x_0)(b-a) > 0 \). This is a contradiction to the fact that \( \int_{a}^{b} f = 0 \).

If \( f(x_0) = 0 \), we are done, so assume \( f(x_0) < 0 \).

If \( f(x_0) < 0 \), then \( f(x) < f(x_0) \) for all \( x \in [a, b] \) as \( \int_{a}^{b} f < f(x_0)(b-a) \).

By part (a). This is a contradiction to \( f(x_0) \geq 0 \).

If \( f(x_0) = 0 \), we are done, so assume \( f(x_1) > 0 \).

We now have \( f(x_0) < 0 < f(x_1) \). Assume \( x_0 < x_1 \), \( x_1 > x_0 \) would be the same. f cont on \([a, b] \Rightarrow f \) cont on \([x_0, x_1]\).

\( c = 0 \) is between \( f(x_0) \) and \( f(x_1) \). By the IVThm (Thm 4.24) there exists a pt: \( x_0 \in (x_0, x_1) \) s.t. \( f(x_0) = c = 0 \).

Note: The statement is that \( x_0 \in [a, b] \). Cannot \( x_0 = a \) or \( x_0 = b \). Not by the last part -- the appl. of IVThm. If \( x_0 = a \) or \( x_0 = b \), then it would be possible. Also if \( x_0 = a \) or \( b \) and \( f(x_0) = 0 \), same \( \boxplus \), then it would be possible.

In either of these cases, the hypo \( \int_{a}^{b} f = 0 \) and some work \( \Rightarrow f(x) = 0 \) for all \( x \in [x_0, x_1] \) (or \([x_1, x_0]\) or \( x \in [a, b] \)) if we knew that either \( x_0 = a \) or \( b \), and \( f(x_0) = 0 \), or \( x_1 = a \) or \( b \) and \( f(x_1) = 0 \).

But it can happen: