Let \( f: [a,b] \to \mathbb{R} \) be continuous and such that \( f(a) < f(b) \).
Let \( c \in (a,b) \). Prove that \( f(a) < f(c) < f(b) \).

**Proof:** Suppose false. Then either \( f(c) > f(b) \) or \( f(c) \leq f(a) \). We'll suppose \( f(c) > f(b) \) (the proof of the other case is very similar).

If \( f(c) = f(b) \) and \( c = b \) (since \( c \in (a,b) \)), then \( f \) is not continuous. So this would be a contradiction.

So we suppose \( f(c) > f(b) \).

By Prop 4.23, \( f \) has an absolute max on \([a,b]\) i.e.
exists \( x_0 \in [a,b] \) s.t. \( f(x_0) \geq f(x) \) for all \( x \in [a,b] \).

Since \( f(c) > f(b) > f(a) \), \( x_0 = a \) or \( x_0 = b \) (and \( f(x_0) \geq f(c) \) but who cares)? but \( f(x_0) > f(b) \).

The picture looks like this:

```
 |   |
|---|---|
a   x_0   c   b
```

The fn. has to go continuously through these 4 pts, hence it can't be 1-1 (will not satisfy the horizontal line test.)

To prove this we choose \( c_0 \) s.t. \( f(b) < c_0 < f(x_0) \).

This is possible since \( f(x_0) > f(b) \).

By the IVT, Thm 4.24 applied to \( f \) on \([x_0,b]\),
exists \( x_1 \in (x_0,b) \) s.t. \( f(x_1) = c_0 \).

Also, \( f(x_0) > c_0 > f(a) \) (because \( f(b) > f(a) \)),

Apply the IVT, Thm 4.24 to \( f \) on \([a,x_0]\), i.e. \( \exists x_2 \in (a,x_0) \)

s.t. \( f(x_2) = c_0 \).

\( f(x_1) = f(x_2) \) but \( x_1 \neq x_2 \) (\( (a,x_0) \cap (x_0,b) = \emptyset \))

This is not continuous. \( \therefore \ f(c) \leq f(b) \).

You might try to prove the other part (assuming \( f(c) \leq f(a) \)) using the same reason.