Suppose \( f: \mathbb{R} \to \mathbb{R} \) is continuous at \( x = x_0 \) and that \( f(x_0) > 0 \).

Prove that there exists an interval \( I = (x_0 - 1/n, x_0 + 1/n) \)
where \( n \in \mathbb{N} \) s.t. \( f(x) > 0 \) for all \( x \in I \).

**Proof:** Hint: See Prop. 4.6. Using the \( \varepsilon-\delta \) formulation of continuity at \( x_0 \) with \( \varepsilon = f(x_0)/2 \), we get a \( \delta \) s.t.
\[
|x-x_0| < \delta \implies \frac{f(x) - f(x_0)}{\delta} < \frac{f(x_0)}{2} \Rightarrow -\frac{f(x)}{\delta} < f(x) - f(x_0) < \frac{f(x)}{\delta} \]

or \( -\frac{f(x_0)}{2} < f(x) < \frac{f(x_0)}{2} \)

Thus, for \( 0 < \delta, \frac{f(x)}{\delta} > \frac{f(x_0)}{2} > 0 \).

Then choose an \( n \in \mathbb{N} \) s.t. \( \frac{1}{n} < \varepsilon \) [Hooray — we get to use the Archimedean Prop.]. Then for \( 1x-x_0| < 1/n \)
\[ \exists x \in (x_0 - 1/n, x_0 + 1/n) \text{ s.t. } f(x) > 0. \]

**Alt. Soln:** Suppose false, i.e., for every \( n \in \mathbb{N} \) \( \exists x \in (x_0 - 1/n, x_0 + 1/n) \text{ s.t. } f(x) \leq 0. \) [That's probably the hard part.]

Let \( n = 1 \), call the "x" value \( x_1 \). Then \( f(x_1) \leq 0 \).

Let \( n = 2 \), call the "x" value \( x_2 \). Then \( f(x_2) \leq 0 \).

Etc.

For each \( n \), we have \( x_n \) s.t. \( f(x_n) \leq 0 \).

Also, \( x_n \in (x_0 - 1/n, x_0 + 1/n) \implies x_n \to x_0. \)

By the cont of \( f \) at \( x_0 \), we know that \( f(x_n) \to f(x_0) \).

Since \( f(x_n) \leq 0 \) for all \( n \), \( f(x_0) \leq 0. \) [This is the "true statement" I gave as a part of the soln to HW 7 (b) — really the equivalent statement for negatives.]

This contradicts the fact that \( f(x_0) > 0 \).