1. Calculate the following limits.
(a) \( \lim_{x \to \infty} e^{-2} \ln x \)

(b) \( \lim_{x \to 0} \frac{1}{x} \ln(1 + 3x) \)

(c) \( \lim_{x \to 1} \frac{x^7 - 1}{x^4 - 1} \)

2. Calculate the following integrals. You must show your work.
(a) \( \int_0^\infty \frac{1}{(3x + 1)^3} \, dx \)

(b) \( \int_{-\infty}^\infty \frac{1}{x^2 + 1} \, dx \)

(c) \( \int_3^5 \frac{1}{\sqrt{x - 2}} \, dx \)

3. (a) Find the 4th order Taylor polynomial approximation of the function \( f(x) = \cos(x) \) about the point \( a = \pi/4 \).

(b) Use Taylor’s Inequality \( |R_n(x)| \leq \frac{M}{(n+1)!} |x - a|^{n+1} \) to estimate the accuracy of the approximation \( f(x) \approx T_4(x) \) when \( x \) satisfies \( 0 \leq x \leq \pi/2 \).

(c) Find the second order Taylor polynomial approximation of the function \( f(x) = \frac{1}{\sqrt{1 - x}} \) about the point \( a = 0 \).

4. Suppose you have derived the 10th order Taylor polynomial plus error term expansion of the function \( f(x) \) about \( a = 0 \) and obtained
\[
f(x) = T_{10}(x) + \frac{1}{10!} \int_0^x (t - x)^{10} f^{(11)}(t) \, dt
\]
where \( T_{10}(x) = \sum_{k=0}^{10} \frac{1}{k!} f^{(k)}(0)x^k \). Use integration by parts to obtain the 11th order Taylor polynomial plus error term expansion of the function \( f(x) \) about \( a = 0 \).

5. Determine whether the following sequences converge or diverge. If the sequence converges, find the limit.
(a) \( a_n = \frac{3 + 5n^2}{n + n^2} \)
(b) \[ a_n = \frac{e^{\ln(n^3) + \ln(n^7)}}{5n^5}. \]

(c) \[ a_n = \frac{(-1)^n}{3n + 1}. \]

(d) \[ a_n = \sin \left( \frac{1}{n^2} \right). \]

6. Solve the following differential equations
(a) \[ x^2(y + 1)^2 \frac{dy}{dx} = 2 \]

(b) \[ \frac{dz}{dt} + e^{t+z} = 0. \]

7. A bacteria culture grows at a rate proportional to its size. The count was 400 after 2 hours and 25,600 after 6 hours.
(a) Write a differential equation that describes the growth of the bacteria culture.
(b) Find an expression for the culture population after \( t \) hours.

(c) What was the initial population?

(d) From some particular time, \( t = t_0 \), how long does it take for the population to double?

8. The definition of \( \lim_{n \to \infty} a_n = L \) is as follows: A sequence \( \{a_n\} \) has the limit \( L \) and we write \( \lim_{n \to \infty} a_n = L \) if for every \( \epsilon > 0 \) there is a corresponding integer \( N \) such that \( |a_n - L| < \epsilon \) whenever \( n > N \).

For \( |r| < 1 \) prove that \( \lim_{n \to \infty} = 0. \)