M161, Test 3, Fall 2008

Name: ________________________________

Section: ________________________________

Instructor: ________________________________

Time: 75 minutes. You may not use calculators on this exam

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<th>Problem</th>
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\[
\frac{d}{dx} \sin(x) = \cos(x), \quad \frac{d}{dx} \cos(x) = -\sin(x), \quad \frac{d}{dx} \tan(x) = \sec^2(x), \\
\frac{d}{dx} \csc(x) = -\csc(x) \cot(x), \quad \frac{d}{dx} \sec(x) = \sec(x) \tan(x), \quad \frac{d}{dx} \cot(x) = -\csc^2(x), \\
\frac{d}{dx} \arcsin(x) = \frac{1}{\sqrt{1-x^2}}, \quad \frac{d}{dx} \arccos(x) = -\frac{1}{\sqrt{1-x^2}}, \quad \frac{d}{dx} \arctan(x) = \frac{1}{1+x^2}, \\
\frac{d}{dx} \arccsc(x) = \frac{1}{x \sqrt{x^2-1}}, \quad \frac{d}{dx} \arcsec(x) = \frac{1}{x \sqrt{x^2-1}}, \quad \frac{d}{dx} \arccot(x) = \frac{1}{1+x^2}, \\
\sin(2x) = 2 \sin(x) \cos(x) \quad \int \ln x \, dx = x \ln x - x + C \quad \int \sec(x) \, dx = \ln|\sec(x) + \tan(x)| + C \\
\tan^2(x) + 1 = \sec^2(x) \quad \cos^2(x) = \frac{1 + \cos(2x)}{2} \quad \sin^2(x) = \frac{1 - \cos(2x)}{2}
\]

Taylor series of the function \( f(x) \) about \( x = a \):

\[
f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n
\]

**Multiple Choice Answer Block**

A  a  b  c  d  e  
B  a  b  c  d  e  
C  a  b  c  d  e  
D  a  b  c  d  e  
E  a  b  c  d  e  
F  a  b  c  d  e
I) Determine, for example by appropriate manipulation of known power series, or by calculating
the Taylor series about $a = 0$, power series for the following functions centered at 0.

Your solution should make clear what steps were performed with series of what functions. If
you use “$+\cdots$” notation, you should give at least the first four nonzero terms of the series to indicate
the pattern of coefficients.

a) $2^x$

b) $f(x) = \int_0^x t \cdot \sin(t) dt.$
c) \( \ln\left(\frac{1 + x}{1 - x}\right) \). (Use logarithm laws to simplify!)
2) Solve the following initial value problem using power series.

\[ y' - x^2 y = 0, \quad y(0) = 1 \]

Identify the function from the power series solution.
3) We want to study the convergence of \( \int_1^{\infty} x^2 \cdot \left( \cos \left( \frac{1}{x^2} \right) - 1 \right) \, dx \), using limit comparison with \( \int_1^{\infty} \frac{1}{x^n} \, dx \).

What choice of \( n \) would you make for the comparison, and why? Explain your argument. 

(Hint: Use the first nonzero term of a Taylor series for \( \cos(t) - 1 \) to see how it behaves for small values of \( t \). In this term set \( t = \frac{1}{x^2} \) and multiply the result by \( x^2 \).)
4) The following multiple choice problems will be graded correct answer only. You do not need
to show work, and no partial credit will be given. Record your answer in the answer block on the
front page. Answers given on these pages will not be scored. You also may tear off these pages and
do not need to hand them in.

It is strongly recommended that you work out the problems until the correct answer is uniquely
determined and don’t just try to solve them by “intuition” or “guessing” – doing so is likely to result
in a wrong pick.

Each correct answer is worth 6 points, each incorrect answer is counted as −2 points. (Unan-
swered questions are 0 points, questions in which more than one answer is ticked are considered to
have been answered wrongly.) You cannot get less than 0 points in this part.

A) Compute the Taylor polynomial of degree 3 about \( a = 0 \) for the function
\[ f(x) = \sin(2x) \]
\[ \begin{align*}
\text{a) } & 2 + x + \frac{x^2}{2} + \frac{x^3}{6} \\
\text{b) } & 2 + x - \frac{x^3}{6} \\
\text{c) } & 2x - 4x^2 + \frac{16x^3}{3} \\
\text{d) } & 2x - \frac{4x^3}{3} \\
\text{e) } & 2x - \frac{x^3}{3}
\end{align*} \]

B) Evaluate the series \( \sum_{n=1}^{\infty} 3^{-n}2^{n+1} \), if the series converges
\[ \begin{align*}
\text{a) } & 2 \\
\text{b) } & 4 \\
\text{c) } & 7 \\
\text{d) } & 9 \\
\text{e) } & \text{The series diverges}
\end{align*} \]
C) What is the coefficient of $x^2$ in the Taylor series for \( \frac{1}{(1 + x)^2} \) about $a = 0$?

- a) \( \frac{1}{6} \)
- b) \( \frac{1}{3} \)
- c) 1
- d) 3
- e) 6

D) A function $f$ has the following Taylor series about $a = 0$:

\[
\frac{x^4}{2!} + \frac{x^5}{3!} + \frac{x^6}{4!} + \cdots + \frac{x^{n+3}}{(n+1)!} + \cdots.
\]

Which of the following is an expression for $f(x)$?

- a) $-3x \sin(x) + 3x^2$
- b) $-\cos(x^2) + 1$
- c) $-x^2 \cos(x) + x^2$
- d) $e^{x^2} - x^2 - 1$
- e) $x^2 e^x - x^3 - x^2$
E) Let $3x^2 - 5x^3 + 7x^4 + 3x^5$ be the fifth-degree Taylor polynomial for the function $f$ about $a = 0$. What is the value of $f'''(0)$?

- $a$ $-30$
- $b$ $-15$
- $c$ $-5$
- $d$ $-\frac{5}{6}$
- $e$ $-\frac{1}{6}$

F) Given $p(x) = \sum_{n=1}^{\infty} \frac{(x-2)^n}{n^2}$, determine the interval, in which $p(x)$ converges.

- $a$ $[1, 3)$
- $b$ $[0, 2)$
- $c$ $[0, 4)$
- $d$ $[-1, 5)$
- $e$ $\mathbb{R}$