M161, Test 3, Fall 2007

Name: __________________________

Section: __________________________

Instructor: __________________________

Time: 75 minutes. You may not use calculators on this exam

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\[
\frac{d}{dx} \sin(x) = \cos(x), \quad \frac{d}{dx} \cos(x) = -\sin(x), \quad \frac{d}{dx} \tan(x) = \sec^2(x),
\]

\[
\frac{d}{dx} \csc(x) = -\csc(x) \cot(x), \quad \frac{d}{dx} \sec(x) = \sec(x) \tan(x), \quad \frac{d}{dx} \cot(x) = -\csc^2(x),
\]

\[
\frac{d}{dx} \sin(x) = \frac{1}{\sqrt{1-x^2}}, \quad \frac{d}{dx} \cos(x) = -\frac{1}{\sqrt{1-x^2}}, \quad \frac{d}{dx} \tan(x) = \frac{1}{1+x^2},
\]

\[
\frac{d}{dx} \acos(x) = -\frac{1}{x\sqrt{x^2-1}}, \quad \frac{d}{dx} \sec(x) = \frac{1}{x\sqrt{x^2-1}}, \quad \frac{d}{dx} \cot(x) = -\frac{1}{1+x^2},
\]

\[
\frac{d}{dx} \acos(x) = \frac{1}{x\sqrt{x^2-1}}, \quad \frac{d}{dx} \sec(x) = \frac{1}{x\sqrt{x^2-1}}, \quad \frac{d}{dx} \cot(x) = \frac{1}{1+x^2},
\]

\[
\int \ln x \, dx = x \ln x - x + C \quad \int \sec(x) \, dx = \ln | \sec(x) + \tan(x) | + C \quad \sin^2(x) = \frac{1-\cos(2x)}{2}
\]

\[
\tan^2(x) + 1 = \sec^2(x)
\]

Taylor series of the function \( f(x) \) about \( x = a \):

\[
f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n
\]

**Theorem** (The Derivative Rule for Inverses) If \( f \) has an interval \( I \) as domain and \( f'(x) \) exists and is never zero on \( I \), then \( f^{-1} \) is differentiable at every point in its domain. The value of \( (f^{-1})' \) at a point \( b = f(a) \) in the domain of \( f^{-1} \) is given by \( (f^{-1})'(b) = \frac{1}{f'(a)} \).
1) a) Determine the value of the following infinite sum:

\[ 8 + 2 + 1/2 + 1/8 + 1/32 + \cdots. \]

b) Consider the series \( \sum_{n=3}^{\infty} \frac{n + 2}{2^n} \). We re-index this series to start at \( n = 0 \), i.e. convert to the form \( \sum_{n=0}^{\infty} a_n \). Determine \( a_n \).
2) a) Calculate the Taylor polynomial for $\ln(\cos(x))$ of degree 2 about $a = 0$.

b) Calculate the Taylor polynomial for $x^4 + x^2 - 2x + 1$ of degree 3 about $a = 1$. 
3) Determine, for example by appropriate manipulation of known power series or by calculating the Taylor series about $a = 0$, power series for the following functions. Your solution should make clear what steps were performed with series of what functions.

   a) $\sin(x^3)$

   b) $\frac{x^2}{1 - x^3}$
c) \( \frac{1}{(1-x)^2} \) (Hint: Find a series for the antiderivative first.)
4) Solve the following initial value problem using power series. (A solution in form of a power series is sufficient, you do not need to recognize the function.)

\[(1 - x)y' - y = 0, \quad y(0) = 1\]