M161, Test 3, Fall 2003

NAME: _______________________

SECTION: ____________________

INSTRUCTOR: __________________

You may use calculators. No formulas can be stored in your calculators.
1. Calculate the following limits.

(a) \[ \lim_{x \to 1} \frac{\ln x}{x - 1} \]

(b) \[ \lim_{x \to 1} \frac{1 - \cos(3x)}{x^3 + 3x} \]

2. Use the integral test to show that the harmonic series diverges.
3. Calculate the following integrals. You must show your work.

(a) \[ \int_{0}^{3} \frac{1}{\sqrt{x}} \, dx \]

(b) \[ \int_{-\infty}^{-1} e^{-2t} \, dt \]
4. (a) Find the 3rd degree (n=3) Taylor polynomial approximation of the function $f(x) = x^{-2}$ about the point $a = 1$.

(b) Use Taylor’s Inequality ($|R_n(x)| \leq \frac{M}{(n+1)!} |x - a|^{n+1}$) to estimate the accuracy of the approximation $f(x) \approx T_3(x)$ when $x$ satisfies $.9 \leq x \leq 1.1$. 
5. Determine whether the series is convergent or divergent. In either case justify your answer.
\[
\sum_{n=1}^{\infty} \frac{n}{n^2 + n + 1}
\]

6. The 3\textsuperscript{rd} order Taylor expansion for the function \( f(x) \) at \( a = 0 \) is given by
\[
f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 - \frac{1}{3!} \int_0^x (t - x)^3 f^{(1)}(t) \, dt.
\]
Find the 4\textsuperscript{th} order Taylor expansion of the function \( f \) about \( a = 0 \) by integration by parts.
7. Determine whether the following sequences converge or diverge. If the sequence converges, find the limit.
(a) \( a_n = \cos\left(\frac{n\pi}{2}\right) \)

(b) \( a_n = \frac{3 + (-1)^n}{5n^2} \)

8. (a) Determine if the sequence \( a_n = 3/(n + 5) \) is increasing or decreasing. Show why or why not.

(b) Is it a monotonic sequence?
9. The rate of change of enrollment at a university is proportional to the enrollment at any time $t$. Suppose there were 20,000 students in 1980 and 30,000 students in 1990.
   (a) Write a differential equations that models the enrollment at the university.

(b) Find an expression for the enrollment at any time $t$ (solve the differential equation and determine the constants).

10. Solve the following differential equation.
   \[ \frac{dy}{dx} = \frac{1 + x}{xy}, \quad x > 0, \quad y(1) = -4 \] (Solve for $y$).
11. The definition of $\lim_{n \to \infty} a_n = L$ is as follows: A sequence $\{a_n\}$ has the limit $L$ and we write $\lim_{n \to \infty} a_n = L$ if for every $\epsilon > 0$ there exists an $N$ so that if $n > N$ then $|a_n - L| < \epsilon$.

(a) Prove that $\lim_{n \to \infty} r^n = 0$ when $-1 < r < 1$ and $r \neq 0$ by showing that for every $\epsilon > 0$ there exists an $N$ so that if $n > N$ then $|r^n - 0| < \epsilon$.

(b) Prove that $\lim_{n \to \infty} r^n = 0$ when $r = 0$ by showing that for every $\epsilon > 0$ there exists an $N$ so that if $n > N$ then $|r^n - 0| < \epsilon$. 
12. Find the sum of the series $\sum_{n=3}^{\infty} \frac{1}{2^n}$.

13. Determine whether the series is absolutely convergent, conditionally convergent or divergent. In any case explain why. You must justify your answers.

(a) $\sum_{n=0}^{\infty} (-1)^n \frac{1}{\sqrt{n + 2}}$

(b) $\sum_{n=1}^{\infty} (-1)^n \frac{n}{e^n}$