Experimental Course: Introduction to Mathematical Reasoning – A Review

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March 26, 2008

Abstract

The course to be discussed is a bridge course that is intended to help undergraduate students make the transition from calculus to the Junior level courses. It was offered in Fall 07 as an experimental course. I will discuss the experiences with that course, the results of a student survey, and the responses I got from an MAA session at the San Diego meeting in January "Crossing the Bridge to Higher Mathematics, What Works and Why" organized by George Davis from Georgia State. I intend to give a short presentation and then open up for discussion.

Problems to Address

Some faculty members perceived a weakness in student's abilities to think abstractly in Junior level classes.

Often, student's knowledge of high school material was sketchy. Problems with

- Proofs, specifically proofs by induction
- Abstraction, for instance the concept of a function being independent of the 'term' of the function. This problem is compounded since functions in Calculus are almost always identified with their term.
- Problems with 'one-to-one' and 'onto.'
- Problems with arithmetic/fractions. Too much reliance on calculators.
- Problems with visualizing.

Problems to Address

This was noticable particularly in Abstract Algebra, but also in Combinatorics, Linear Algebra, Cryptography etc.

The problem was most critical in Abstract Algebra (MATH366), where we felt that the course could no longer achieve its goals.

Since MATH366 is the only algebra course that educators take, the question was also:

> *How much Algebra / Pure Math do we teach to a future high school teacher?*

We felt that our current course offering is not optimal (not only for educators).

Situation

We suggested to offer a transition course "Introduction to Mathematical Reasoning"

Offered as experimental 3-credit course in Fall 2007.

Student evaluation done, we wish to get back to the same students again in a year or so.

Goals

Ease the transition to Junior level courses, and familiarize students with the concept of Proof.

The course should be taken by Sophomores (or even Freshmen), and complement some of the material in CALCIII (analytic geometry).

It should have content.
It is not at the level of 'What is Mathematics?' or 'What is Mathematics, really?' (even though the book by Courant 'What is Mathematics?' was used and turned out to be very valuable).

Goals

It should be broad. In particular, concepts from analysis are included. It is not a 'Discrete Math' course that talks exclusively about induction, functions and logic.

It sould also advertise the different course offerings at the Junior level by touching upon the various topics briefly.

Mathematics of the infinite was not a major component. We felt that students were not ready to appreciate some of the more subtle points. Nevertheless, the fact that countable by countable is countable was touched upon.

Place in the Curriculum

Prerequisites: Calculus I (to exclude some audiences with little preparation)

The transition course could also serve a selection role for the two algebra tracks now offered: MATH366 or MATH466/467. It would allow the students to "sort" themselves into one of the two tracks.

The Textbook(s)

Books used (in the order most used down to least used):

- Courant, Robbins, Stewart: What is Mathematics?
- Liebeck: A Concise Introduction to Pure Mathematics (the official book for the course).
- Lambacher, Schweizer: Analytische Geometrie (German High School text covering analytic geometry).
- Velleman: How to prove it: A structured approach.

The Syllabus

The course had three 5-week blocks (with midterms in week 5 and 10).

The 1st block followed mostly Courant's book up until complex numbers (including logic, proof techniques, the number system, induction, inequalities).

The 2nd block followed mostly Liebeck's book (including the number theory parts) plus pieces of Velleman's book for one-to-one and onto and induction.

The 3rd block followed mostly Lambacher, Schweizer for analytic geometry. Also touched: Cryptography and Combinatorics.

The Syllabus: 1st Block: Week 1-5

Week 1: Talk about "famous" mathematical results, what is mathematics, statements, contraposite. Euclid: infinitely many primes, representation of integers, logic.
Week 2: Computing in different bases, what are numbers, why is $\sqrt{2}$ irrational, approximation of real numbers by rationals, $\sqrt{2} \approx 1.4$ decimal fractions, periodic expansion $\iff$ rational, definition of limit, algebraic vs. transcendental, quadratic formula, what if $b^2 - 4ac < 0$?, complex numbers.
Week 3: Inequalities, Complex numbers, Argand/Wessel diagram, polar vs cartesian, addition / multiplication, DeMoivre, roots of unity
Week 4: More on complex numbers, $e^{i\theta}$ notation, solving $z^3 = 1 + i$ (for instance)
Week 5: Review, Midterm 1, correction

The Syllabus: 2nd Block: Week 6-10

Week 6: $\sum$ and $\prod$ notation, induction, Fibonacci numbers, functions, one-to-one, onto, inverse function, unusual functions.
Week 7: Continue induction, Pigeonhole principle, Euclid's algorithm.
Week 8: Extended Euclidean algorithm, simple divisibility facts, elementary number theory, unique prime factorization, modular arithmetic
Week 9: solving congruences, repeated squaring and multiplying, test of 3, solving congruences $ax \equiv b \bmod n$ (mostly with $\gcd(a, n) = 1$) inverse mod n,
Week 10: Review, Midterm 2, correction.

The Syllabus: 3rd Block: Week 10-15

Week 10: Cryptography, Caesar cipher, shift cipher, affine cipher $ax + b$, test of 9, test of 11, review induction, review one-to-one / onto.
Week 11: Analytic geometry, $\mathbb{R}^2$, equations of lines (6 different types)
Week 12: More on vectors, parametric form of lines. Hesse normal form, dot product, orthogonality, distance point / line, lines at an angle.
Week 13: More analytic geometry, area computations, $\mathbb{R}^3$, lines and planes.
Week 14: Combinatorics: binomial coefficients, recurrence relation vs. direct formula, proof of $(x+y)^n = \sum_{i=0}^n \binom{n}{i} x^i y^{n-i}$ by induction.
Week 15: Review

Samples

The coconut problem: Five suspicious sailors are stranded on a remote island. They spend the day gathering a pile of coconuts. Exhausted, they postpone dividing it until the next morning. Suspicious, each decides to take share during the night. The first sailor divides the pile into five equal portions plus one extra coconut, which he gives to a monkey. He takes one pile and leaves the rest in a single pile. The second sailor later does the same; again the monkey receives one leftover coconut. The third, fourth and fifth sailor also do this; each time a remainder of one goes to the monkey. In the morning, they split the remaining coconuts into five equal piles, and each gets his "share". Each knows some were taken, but none complains, since each is guilty!) What is the smallest possible number of coconuts in the original pile?

Samples

The Zebra Problem: There are five houses in a row, each of a different color, inhabitated by women of different nationalities. The owner of each house owns a different pet, serves different drinks, and smokes different cigarettes from the other owners. The following facts are also known:
The Englishwomen lives in the red house.
The Spaniard owns a dog.
Coffee is drunk in the green house.
The Ukrainian drinks tea.
The green house is immediately to the right of the ivory house.
The Oldgold smoker owns the snail.
Kools are smoked in the yellow house.
Milk is drunk in the middle house.
The Norwegian lives in the first house on the left.
The Chersterfield smoker lives next to the fox owner.
The yellow house is next to the horse owner.
The Lucky Strike smoker drinks orange juice.
The Japanese smokes Parliament.
The Norwegian lives next to the blue house.

The question: Who drinks water and who owns the zebra?

Samples

Lewis Carroll: Animals, that do not kick, are always unexcitable. Donkeys have no horns. A buffalo can always toss one over a gate. No animals that kick are easy to swallow. No hornless animal can toss one over a gate. All animals are excitable, except buffaloes. Therefore, donkeys are not easy to swallow.

Show that this is logical!

Evaluate the following quantity: $\prod_{j=1}^{4} \sum_{i=0}^{3} 3j + 2i.$

Negate the following statement: $n \geq 7$
Answer: $n < 7$ or $n \leq 6$ (if n is an integer)

Samples

If an undergraduate club is supposed to include at least 7 freshmen or at least 6 sophomores or at least 5 juniors or at least 4 seniors, what is the least number of students who are needed to join to meet the condition regardless of how the students are selected.

Prove that if (a, b, c) is a Pythagorean triple, then ab is even. Prove that if (a, b, c) is a primitive Pythagorean triple, then a and b are of opposite parity.

Samples

Find the integer coefficients a, b, c, d so that the following Chemical reaction formula makes sense:

$$aC_2H_6 + bO_2 \quad \rightarrow \quad cCO_2 + dH_2O.$$

Let $f : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$ (with $\mathbb{N} = \{0, 1, 2, \ldots\}$) be given by

$$f(i,j) = \frac{(i+j+1)(i+j)}{2} + j.$$

a) Is f one-to-one?
b) Is f onto?

Answer: yes to both (this is the argument why countable $\times$ countable $=$ countable)

Samples

Prove by induction:

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots+\frac{1}{2n-1}-\frac{1}{2n}=\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{2n}$$

Student's Evaluation

Students found the course hard – actually very hard. This was an issue particularly in the beginnig.

Students were unsatisfied with the textbook (Liebeck). This was probably because the textbook was a little too tough on them. Also, the textbook used the quantifier notations which threw students off.

The lecture did not closely follow the textbook which confused some students.

The Survey

We have the results of 10 Students (enrollment was 21)

1) a) List the Calculus courses already taken:
Calc I: 1, Calc II: 1, Calc III: 8

b)List any upper division courses you have taken:
Statistics: 2, ODEs: 3, Linear Algebra: 1, Math Modeling: 1

The Survey

2) a)When you registered for the course, did you have any specific expectations?
yes: 3, no: 10
If yes, please describe what you expected the course would be like:

- ☐ expected it to be fun and somewhat challenging, and it was (A yes)
- ☐ Learning and practicing proof techniques (A: yes)
- ☐ I expected lots of proofs and be able to do them confidently (A: yes)
- ☐ My advisor informed me that it would relate math to real life. I wasn't sure what that meant but I didn't expect to do proofs and stuff like that. Now that I look back at it the title of the course makes sense. (A: no)

b)Did the course meet your expectations:
fully: 1, partly: 6,no: 0

The Survey

3) Did the course influence your interest in mathematics?
much more interested: 0, more interested: 5, no change: 4, less interested: 1, much less interested: 0

Comments:

- ☐ Compared to my calculus course, this was much more structured and the material was covered in a more formal manner. I think to know why certain things occur is more fascinating than just learning how to "plug and chug" (A: MI)
- ☐ This course made me more confused about math on most subjects but helped me have a better understanding on a couple of subjects (A: N)
- ☐ I was already very interested in Math, this course brought up some things I have never learned and make we want to learn more about Math. However I feel like this class sort of wasted my time in the sense that it did not have focus and just merely introduced ideas and then dropped them so I did not fully learn anything (A: MI)
- ☐ I am a Math major, so I was already interested (A: MI)

The Survey

- ☐ I have already been interested in Mathematics because there were concrete methods that I could follow. These proofs really confused me because it seemed like we would pull things out of thin air to get what we needed. Every proof was always different so it was hard to look back on past examples to get a credible answer (A: LI)
- ☐ I want to be a math teacher, so it is important for me to know proofs, but the presentation of information was fairly scattered and hard to follow. (A: N)
- ☐ Different than what I'm used to (A: N)
- ☐ I'm a math major already (A: N)
- ☐ Seeing why certain equations work is really interesting, even though I have issues trying to figure them out sometimes (A: MI)

The Survey

4) One of the goals of the course is to familiarize students with the concept of mathematical proof. Do you agree that you achieved this goal? strongly agree: 2, agree: 5, neutral: 2, disagree: 1, strongly disagree: 0

The Survey

5) a) In your opinion, is this course different from the calculus courses you have taken?
very different: 5, somewhat different: 5, similar: 0

- ☐ In my calculus classes, we hardly ever learned why things work out the way they do. In here I finally know why and how (A: VD)
- ☐ Just that it seemed so abstract when compared to calculus courses. There were no equations to follow or anything like that (A: VD)
- ☐ Some concepts I learned in this class helped my calculus class be easier to understand (A: SD)
- ☐ The other calculus classes I've taken were less proof based than this one was (A: SD)

b) How much time did you invest in this course compared to calculus courses?
much more: 1, slightly more: 1, same: 3, slightly less: 4, much less: 1

The Survey

6) Do you agree that this course was interesting?
strongly agree: 0, agree: 8, neutral: 1, disagree: 1, strongly disagree: 0

7) Do you agree that this course was difficult:
strongly agree: 5, agree: 2, neutral: 2, disagree: 1, strongly disagree: 0

8) Would you recommend this course to a friend who is a beginning Math major?
definitely: 2, probably: 3, not sure: 3, probably not: 2, certainly not: 0

The Survey

9) Please describe the strengths and weaknesses of the course:

Strengths:

- ☐ Learning the different types of proofs was very helpful. I think when I go to my analysis class I will have some background instead of walking in cold.
- ☐ The professor was very helpful, friendly and approachable.
- ☐ It taught mathematical reasoning better than other courses that often seem more mechanistic.
- ☐ I think this course did a good job of introducing many new topics and familiarize people who have never learned proofs with general ideas.
- ☐ Makes you think what you are trying to accomplish.
- ☐ Interesting, good information. A good catch-all course. Professor is good for this course.

The Survey

- ☐ The teacher makes it interesting. A class much different that anything I've taken – very fun.
- ☐ I thought this class was more interesting than a calculus course. I liked how it was more logical and we had to use our brain differently. I really like the teacher, whenever I went for help, he was eager to teach me.
- ☐ Seeing so many proofs worked out helps see how they are working more clearly.
- ☐ Covers many topics, different proof types.

The Survey

Weaknesses:

- ☐ I think that maybe this class should cover a variety of intros into the upper division classes so we get an idea as to what interests us and what doesn't.
- ☐ The profesor expected us to have a lot of knowledge about certain math concepts that I had no exposure to and I got lost often.
- ☐ For a three credit course, I think it would have moved faster and covered more material.
- ☐ I noticed there were a lot of confused people. They did not push us enough to think of our own. It did not follow a book and thus was hard to learn on my own and I do not always think or learn how teachers teach. It was too much information and not enough detail about each topic. Each type of problem was often times only shown once on homework or in class and then we were expected to do it on tests.
- ☐ Fact paced for the introduction of new thought. The text book was not very helpful.

The Survey

☐ The organization was hard to follow. Rather than learning proofs, then discovering other things through that, there was a wide range of topics, presented, then proof-writing, thrown in. Textbook was not helpful.
☐ The lessons sometimes don't correlate with the homework. Class material is hard.
☐ I think at times what we did in class didn't have much to do with the homework. We were left at times unable to do the homework and we would bring lots of questions to class and he would help us.
☐ Because the course is brand new, not all the kinks are quite worked out, like where to find extra help not from the teacher or the course book.
☐ The text doesn't seem very helpful.

Other Feedback

I also received feedback from a student (Adam Wissemann, honors) through a different channel:

> *He [the teacher] was extremely effective in teaching me proof techniques ... that I have found to be invaluable in a couple of my current math classes.*

continued...

Other Feedback

> *Learning about a very wide variety of topics within math [was my favorite part of being in the class]. The class was intended to be primarily a transition course between lower level math classes (Calc 1,2,3, etc., in which students just compute things) and upper level classes in which proofs are very important. In this respect Dr. Betten was free to jump from topic to topic as he saw fit. While some students in the class did not appreciate this, I found it to be a lot of fun. There is more to math than Calculus, or Differential Equations, and I enjoyed learning about a wide variety of topics. In particular, his lectures on encryption piqued my interest and made me interested in taking Mathematics of Information Security next year.*

Teacher's Comments

☐ I noticed the homework was very bad in the beginning. After two weeks or so I could see an improvement.

☐ I was surprised how hard induction and one-to-one/ onto is for the students.

☐ I noticed that there were a lot of students who struggled with induction even in the second midterm. I took the third part of the course to really enforce induction.

Teacher's Comments

☐ In my other courses, I sometimes do student's presentations in the end. Some of them are nice but some students really have difficulties expressing themselves in a manner that makes sense to the audience.
I am wondering if 'Mathematical Communication' should be moved up in the list of goals that we try to achieve with our curriculum.
Could this be a topics for the bridge course or should we devote some other space for such a topic?

The San Diego Meeting

I was invited to join an MAA session 'Crossing the Bridge to Higher Mathematics – What Works and Why' at the San Diego Meeting January 2008.

I was surprised to see how many people try to address the same issues and have similar courses.

I received a lot of feedback. Noticable was the encouragement from Charles Matthews (Chair at Southeastern Oklahoma State), who told me he started like that some 15 years ago and encouraged me to continue.
In later communication, he offered me to use his text in my class free of charge (under the condition that I would give him feedback and comments).

The San Diego Meeting

A lot of people talked about including a writing component ('portfolio') into their courses:
Penny Dunham / Muhlenberg College
Magnhild Lien / California State University at Northridge