There is No Drake / Larson Linear Space on 30 Points

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Abstract

A linear space is Drake / Larson if it contains at least two lines and there are no lines of size $2, 3$ or 6. The existence or nonexistence of such linear spaces on v points is known except for $v = 30$. The purpose of this paper is to settle the remaining case on thirty points in the negative. This result relies on a combination of parameter calculation and exhaustive computer search.

1 Introduction and Statement of Results

A linear space (cf. [1]) is an incidence structure of points and lines in which each line consists of two or more points and any two points are contained in exactly one line. A linear space is called Drake / Larson if it contains at least two lines and the size of no line divides six. The existence or nonexistence of such linear spaces on v points is decided in [6] for all $v \neq 30$ (see also [7]). We prove:

Theorem 1 *There is no Drake / Larson linear space on* 30 *points.*

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Case	Line Type	Comment
1	$8^1, 7^1, 5^{14}, 4^{41}$	[8]
2	$7^3, 5^{24}, 4^{22}$	[2]
3	$7^3, 5^{15}, 4^{37}$	Section 4
4	$7^1, 5^{27}, 4^{24}$	Section 5
5	$7^1, 5^{24}, 4^{29}$	Section 6
6	$7^1, 5^{15}, 4^{44}$	Section 7

Table 1: Possible Line Types for a Drake / Larson Linear Space on 30 Points

We remark that the introduction of [8] contains a description of the repercussions of this result.

Let $\mathcal{S} = (\mathcal{V}, \mathcal{L})$ be a finite linear space with point set $\mathcal{V}$ and line set $\mathcal{L}$. Let a_i be the number of lines of size i in $\mathcal{L}$. The vector $(a_2, a_3, \ldots, a_v)$ is the *line type* of $\mathcal{S}$. Often we will use the shorthand notation $v^{a_v}, \ldots, 3^{a_3}, 2^{a_2}$ to denote this type (and omit terms with $a_i = 0$). From earlier work of Drake and Larson [7] it is known that the line type of a Drake / Larson linear space $\mathcal{S}$ on 30 points is one of six cases, listed in Tab. 1.

The fact that Case 1 is not realizable was shown in [8]. Case 2 has been settled in the negative by [2]. Hence it suffices to eliminate the remaining cases $3, 4, 5$ and 6.

Let us describe the plan of this paper. In Section 2 we will recall the concept of a tactical decomposition of an incidence structure. In Section 3 we present a method to synthesize TDO for linear spaces with a given line type. This method will then be applied in Sections 4, 5, 6 and 7 to each of the open cases of line types 3-6 for Drake / Larson linear spaces on 30 points. The resulting parameter cases are then eliminated by computer search. This will complete the proof of Theorem 1.

The final step of our proof involves an exhaustive computer search that required substantial computing efforts (roughly five years CPU-time). The details about this search can be found at our website

http://www.math.colostate.edu/~betten/DL/drake_larson.html

We wish to point out that the authors have performed two independent computer searches, with two completely different implementations of the search algorithm. In both cases, the programs came up with the same (nonexistence) result.

A short while after this paper was first submitted, we received note from Clement Lam that he and his collaborators Ron Mullin and Narges Simjour have independently proved Theorem 1 also.

2 Tactical Decompositions

Let us recall the concept of a tactical decomposition of a linear space as introduced for instance in [5]. A *decomposition* of a linear space $\mathcal{S} = (\mathcal{V}, \mathcal{L})$ is a pair $(\mathfrak{V}, \mathfrak{B})$ of ordered partitions $\mathfrak{V} = (V_1, \ldots, V_m)$ of points and $\mathfrak{B} = (B_1, \ldots, B_n)$ of lines. For a point $p \in \mathcal{V}$, let (p) be the set of lines $L \in \mathcal{L}$ such that $p \in L$.

A decomposition $(\mathfrak{V}, \mathfrak{B})$ of $\mathcal{S}$ is said to be *row-tactical* (or *point-tactical*) if for each $V \in \mathfrak{V}$ and each $B \in \mathfrak{B}$ the number

$$|(p) \cap B|$$

is independent of the choice of the point $p \in V$.

A decomposition $(\mathfrak{V}, \mathfrak{B})$ of $\mathcal{S}$ is said to be *column-tactical* (or *line-tactical*) if for each $V \in \mathfrak{V}$ and each $B \in \mathfrak{B}$ the number

$$|V \cap \ell|$$

is independent of the choice of the line $\ell \in B$.

A decomposition $(\mathfrak{V}, \mathfrak{B})$ of $\mathcal{S}$ is *tactical* if it is both point-tactical and line-tactical with respect to $\mathcal{S}$.

We note that every linear space admits a tactical decomposition, as for example the discrete partition of points and lines always has this property (we call this the *discrete decomposition*). However, for the purposes of this paper, the discrete decomposition is almost never of interest.

A row-tactical decomposition $(\mathfrak{V}, \mathfrak{B})$ of a linear space gives rise to a certain set of combinatorial numbers (a.k.a. structure constants) that we call *decomposition scheme.* These numbers are $v_i = |V_i|$ for $i = 1, \ldots, m$ $b_j = |B_j|$ for $j = 1, \ldots, n$ together with the integers $r_{V,B} = |(p) \cap B|$ where p is an arbitrary point of the point class $V \in \mathfrak{V}$ and B is a line class in $\mathfrak{B}$. More specifically, if $\mathfrak{V} = (V_1, \ldots, V_m)$ and $\mathfrak{B} = (B_1, \ldots, B_n)$, we write $r_{i,j}$ for r_{V_i,B_j}. We agree to display such a scheme in the form of an array:

$$\begin{array}{c|cccc} \rightarrow & b_1 & b_2 & \cdots & b_n \\ \hline v_1 & r_{1,1} & r_{1,2} & \cdots & r_{1,n} \\ \vdots & \vdots & & & \vdots \\ v_m & r_{m,1} & r_{m,2} & \cdots & r_{m,n} \end{array} \tag{1}$$

Here, the horizontal arrow in the upper left corner will remind us of the fact that this scheme describes a row-tactical decomposition.

In a similar fashion, if we are given a column-tactical decomposition $(\mathfrak{V}, \mathfrak{B})$ of a linear space, we define $k_{V,B} = |V \cap \ell|$ where ℓ is an arbitrary line of the line class $B \in \mathfrak{B}$ and V is a point class in $\mathfrak{V}$. In the same way as before, if $\mathfrak{V} = (V_1, \ldots, V_m)$ and $\mathfrak{B} = (B_1, \ldots, B_n)$, we write $k_{i,j}$ for k_{V_i,B_j} and we display the scheme in a likewise manner:

$$\begin{array}{c|cccc} \downarrow & b_1 & b_2 & \cdots & b_n \\ \hline v_1 & k_{1,1} & k_{1,2} & \cdots & k_{1,n} \\ \vdots & \vdots & & & \vdots \\ v_m & k_{m,1} & k_{m,2} & \cdots & k_{m,n} \end{array} \tag{2}$$

Here, the downward arrow in the upper left corner will remind us of the fact that this scheme describes a column-tactical decomposition.

We remark that a tactical decomposition $(\mathfrak{V}, \mathfrak{B})$ as above gives rise to the set of $m \times n$ well-known equations

$$v_i r_{i,j} = k_{i,j} b_j \quad \text{for } 1 \le i \le m, \quad 1 \le j \le n.$$

The line type as introduced above corresponds to a column-tactical decomposition. Later, we will see how to obtain more detailed information using higher order tactical decompositions.

Let us now consider the case that we are given a set of structure constants

$$v_1, \ldots, v_m, b_1, \ldots, b_n, r_{1,1}, \ldots, r_{m,n}.$$

We wish to decide whether there is a linear space $\mathcal{S} = (\mathcal{V}, \mathcal{L})$ that admits a row-tactical decomposition $(\mathfrak{V}, \mathfrak{B})$ with $\mathfrak{V} = (V_1, \ldots, V_m)$ and $\mathfrak{B} = (B_1, \ldots, B_n)$ such that

1. $|V_i| = v_i$ for $i \in \mathbb{Z}_m$,
2. $|B_j| = b_j$ for $j \in \mathbb{Z}_n$,
3. $r_{i,j} = |(p) \cap B_j|$ for each $p \in V_i$ for $i \in \mathbb{Z}_m$ and for $j \in \mathbb{Z}_n$.

If such a linear space does indeed exist, we call the decomposition scheme (1) *realizable.*

In a similar fashion, we may wish to consider structure constants

$$v_1, \ldots, v_m, b_1, \ldots, b_n, k_{1,1}, \ldots, k_{m,n}.$$

If there is a linear space $\mathcal{S} = (\mathcal{V}, \mathcal{L})$ that admits a column-tactical decomposition $(\mathfrak{V}, \mathfrak{B})$ with $\mathfrak{V} = (V_1, \ldots, V_m)$ and $\mathfrak{B} = (B_1, \ldots, B_n)$ such that

1. $|V_i| = v_i$ for $i \in \mathbb{Z}_m$,
2. $|B_j| = b_j$ for $j = 1 \in \mathbb{Z}_n$,
3. $k_{i,j} = |V_i \cap \ell|$ for each $\ell \in B_j$, for $i \in \mathbb{Z}_m$ and for $j \in \mathbb{Z}_n$.

we say that the column-tactical decomposition scheme (1) is *realizable.*

Let us fix some more notation related to partitions of a set. At first, the unit partition of a set X is denoted as $\mathfrak{I}_X$. It has exactly one class consisting of the set X, i.e., $\mathfrak{I}_X = (X)$. Next, the well-known ordering of partitions is as follows. For two partitions $\mathfrak{A}$ and $\mathfrak{B}$ we write $\mathfrak{A} \preceq \mathfrak{B}$ if $\mathfrak{A}$ is a refinement of $\mathfrak{B}$. This means that each class of $\mathfrak{B}$ can be written as a union of classes of $\mathfrak{A}$ (including the possibility that $\mathfrak{A} = \mathfrak{B}$). This ordering applies to both ordered and unordered partitions.

It is well-known that this ordering induces a lattice structure on the set of (unordered) partitions. That is, for any two partitions $\mathfrak{P}$ and $\mathfrak{Q}$, there is a coarsest common refinement $\mathfrak{P} \wedge \mathfrak{Q}$ (greatest lower bound) and a finest partition that is coarser than the two given partitions, denoted $\mathfrak{P} \vee \mathfrak{Q}$ (least upper bound). We introduce the following notation: If $\mathfrak{A} \preceq \mathfrak{B}$ and $A \in \mathfrak{A}$, denote by $\mathfrak{B}(A)$ the class $B \in \mathfrak{B}$ with $A \subseteq B$. This class is also called *ancestor class* of A with respect to $\mathfrak{B}$.

Lemma 2 *Let $\mathcal{S} = (\mathcal{V}, \mathcal{L})$ be a finite incidence structure.*

1. *Let $(\mathfrak{V}, \mathfrak{B})$ be a row-tactical decomposition of $\mathcal{S}$. Then there exists a decomposition $(\mathfrak{V}, \mathfrak{B}')$ with the properties*
   (a) $\mathfrak{B}' \preceq \mathfrak{B}$
   (b) $(\mathfrak{V}, \mathfrak{B}')$ *is column-tactical.*
   (c) *If $(\mathfrak{V}, \mathfrak{C})$ is any column-tactical decomposition of $\mathcal{S}$ then $\mathfrak{C} \preceq \mathfrak{B}'$.*

   Moreover, the partition $\mathfrak{B}'$ is unique up to reordering of its classes.

2. *Let $(\mathfrak{V}, \mathfrak{B})$ be a column-tactical decomposition of $\mathcal{S}$. Then there exists a decomposition $(\mathfrak{V}', \mathfrak{B})$ with the properties*
   (a) $\mathfrak{V}' \preceq \mathfrak{V}$
   (b) $(\mathfrak{V}', \mathfrak{B})$ *is row-tactical.*
   (c) *If $(\mathfrak{C}, \mathfrak{B})$ is any row-tactical decomposition of $\mathcal{S}$ then $\mathfrak{C} \preceq \mathfrak{V}'$.*

   Moreover, the partition $\mathfrak{V}'$ is unique up to reordering of its classes.

Proof. It is easy to verify that $(\mathfrak{V}, \mathfrak{B}')$ with

$$\mathfrak{B}' = \bigvee_{\substack{\mathfrak{C} \preceq \mathfrak{B} \\ (\mathfrak{V}, \mathfrak{C}) \text{ col.tact.}}} \mathfrak{C}$$

has the required properties (note that any discrete partition of the columns induces a column-tactical decomposition and hence the expression is not empty). The other part of the statement follows by considering the dual incidence structure. □

The refinements of the previous lemma are called *coarsest row-tactical refinement* and *coarsest column-tactical refinement*, respectively. The coarsest row-tactical / column-tactical refinement is unique up to ordering of classes with equal ancestor. The lexicographic order can be used as a tie-breaker. We agree to arrange the classes with equal ancestor in such a way that the structure constants are lexicographically decreasing. If the refinement has this property, we call it the *canonical coarsest row-tactical (or column tactical) refinement.*

The refinement procedure for tactical decompositions defined above may be repeated. This works as follows: A column tactical decomposition is refined to a point tactical refinement, this refinement in turn is refined to another column tcatical refinement, which in turn is refined again. The process stops once a tactical decomposition is reached (i.e., a decomposition that is both row-tactical and column-tactical). The resulting sequence of refining decompositions is called *decomposition stack*. It consists of partitions that are strictly refining each other and that are alternately row-tactical and column-tactical.

We wish to consider decomposition stacks satisfying the following two properties:

1. Each decomposition is the canonical coarsest row/column-tactical refinement of its predecessor.

2. The last decomposition is tactical.

Such a canonical decomposition stack is called *tactical decomposition by ordering*, or *TDO*, for short. The concept of a TDO is due to D. Betten and M. Braun [4]. The length of the sequence of decompositions is called the *TDO-depth*. A linear space whose TDO-depth is one is called regular in [3].

Let $(\mathfrak{V}, \mathfrak{B})$ be a row-tactical decomposition of the linear space $\mathcal{S} = (\mathcal{P}, \mathcal{L})$. Let $\mathfrak{V} = (V_1, \ldots, V_m)$ and $\mathfrak{B} = (B_1, \ldots, B_n)$. Write $v_i = |V_i|$ $(i \in \mathbb{Z}_m)$ and $b_j = |B_j|$ $(j \in \mathbb{Z}_n)$. Let $r_{i,j}$ be the associated structure constants.

Lemma 3 *Let $V_{i_1}, V_{i_2}, \ldots, V_{i_s}$ be a subset of classes of $\mathfrak{V}$. For $j = 1, \ldots, n$, define $w_j = \sum_{u=1}^{s} r_{i_u,j} v_{i_u}$ and write $w_j = f_j b_j + e_j$ with $0 \le e_j < b_j$. Assume that*

$$\sum_{j=1}^{n} \left\{ e_j \binom{f_j + 1}{2} + (b_j - e_j) \binom{f_j}{2} \right\} > \binom{\sum_{u=1}^{s} v_{i_u}}{2}.$$

Then the decomposition scheme is not realizable.

Proof. Consider the number of pairs of points taken from $X = \bigcup_{u=1}^{s} V_{i_u}$ that are covered by lines in a linear space with the given row-tactical decomposition. The number of incidences between points from X and lines from B_j is $w_j = \sum_{u=1}^{s} r_{i_u,j} v_{i_u}$. The lines from B_j cover at least $e_j \binom{f_j+1}{2} + (b_j - e_j) \binom{f_j}{2}$ pairs of points from X. Since each pair of points from X can be covered once only, the decomposition is not realizable. □

3 Synthesizing Decomposition Schemes

In order to use the TDO invariants for classification purposes of linear spaces, we wish to describe a procedure that enables us to *synthesize* TDO of arbitrary depth, starting from initial parameters like the line type. Once all TDO of a given depth have been synthesized, the geometric test of Lemma3 is applied to rule out cases that cannot be realized. The remaining TDO are then handed over to a computer program that finds all realizations of a given TDO or proves that no realization exists. In this section, we wish to describe the algebraic process of synthesizing TDO for linear spaces from the line type. Here, a line type is simply a list of integers $(a_v, \dots, a_2)$ satisfying the equation

$$\sum_{i=2}^{v} a_i \binom{i}{2} = \binom{v}{2}.$$

Suppose that $v^{a_v}, \dots, 2^{a_2}$ is a line type. We wish to compute all possible TDO for (putative) linear spaces with this line type.

In the following, we will use $P(n, k)$ to denote the largest number of k-lines in a linear space on n points. These numbers are the *packing numbers* for linear spaces. For our purposes, upper bounds for $P(n, k)$ like the first and second Johnson bound are sufficient (cf. [9, VI.40.7 and VI.40.9]).

3.1 The First Row-Refinement

Let $v^{a_v}, \dots, 2^{a_2}$ be a line type for a putative linear space on v points. We introduce nonnegative integer variables $x_v, \dots, x_2$. Then we solve

$$\sum_{j=2}^{v} (j-1)x_j = v - 1 \tag{3}$$

subject to the conditions $x_j \leq a_j$ for $j \in \mathbb{Z}_{[2,v]}$. Let U be the number of solutions, and let $\mathbf{x}^{(u)} = (x_v^{(u)}, \dots, x_2^{(u)})$ for $u \in \mathbb{Z}_U$ be the u-th solution $(x_v, \dots, x_2)$. We introduce nonnegative integer variables $y_1, \dots, y_U$. Then we solve

$$\sum_{u=1}^{U} x_j^{(u)} y_u = j a_j, \quad \text{for all } j \in \mathbb{Z}_{[2,v]}, \tag{4}$$

and

$$\sum_{u=1}^{U} \binom{x_j^{(u)}}{2} y_u \leq \binom{a_j}{2}, \quad \text{for all } j \in \mathbb{Z}_{[2,v]}, \tag{5}$$

and

$$\sum_{u=1}^{U} x_{j_1}^{(u)} x_{j_2}^{(u)} y_u \leq a_{j_1} a_{j_2}, \quad \text{for all } j_1, j_2 \in \mathbb{Z}_{[2,v]}, j_1 \neq j_2, \tag{6}$$

and

$$\sum_{u=1}^{U} y_u = v, \tag{7}$$

and

$$\sum_{\substack{u=1 \\ x_j^{(u)} \geq t}}^{U} y_u \leq P(a_j, t), \quad \text{for all } j \in \mathbb{Z}_{[2,v]}, \text{ for all } t \in \mathbb{Z}_{[3,a_j]}. \tag{8}$$

Let $y_1, \ldots, y_U$ be a nonnegative integer solution to these systems of equations and inequalities. The row-scheme

$$\begin{array}{c|cccc} \to & a_v & a_{v-1} & \cdots & a_2 \\ \hline y_1 & x_v^{(1)} & x_{v-1}^{(1)} & \cdots & x_2^{(1)} \\ \vdots & \vdots & & & \vdots \\ y_U & x_v^{(U)} & x_{v-1}^{(U)} & \cdots & x_2^{(U)} \end{array} \tag{9}$$

is said to be obtained from the line type $v^{a_v}, \ldots, 2^{a_2}$ on v points by *row-refinement of the first kind.*

Lemma 4 *Let $\mathcal{S}$ be a linear space with line type $v^{a_v}, \ldots, 2^{a_2}$. The coarsest row-tactical refinement of $\mathcal{S}$ is a row-refinement of the first kind of the given line type.*

Proof. Let $\mathcal{S} = (\mathcal{V}, \mathcal{L})$ be a linear space with a_j lines of size j for $j \in \mathbb{Z}_{[2,v]}$. Let $\mathcal{L}_j$ be the set of lines of size j.

Let p be a point in $\mathcal{V}$. Let $x_j = |(p) \cap \mathcal{L}_j|$. Double counting the set of pairs (q, ℓ) with $q \in \mathcal{V} \setminus \{p\}$ and $\ell \in \mathcal{L}$ such that p and q are both on ℓ yields (3). This shows that for any point $p \in \mathcal{V}$, the point type $(x_v, \ldots, x_2)$ is a solutions to (3) and hence there exists an u with $1 \leq u \leq U$ such that

$(x_v, \ldots, x_2) = \mathbf{x}^{(u)}$. We let y_u denote the number of points $p \in \mathcal{V}$ such that the point type of p equals $\mathbf{x}^{(u)}$. The vector $(y_1, \ldots, y_U)$ is associated to the linear space $\mathcal{S}$ and depends only on the ordering of the solutions $\mathbf{x}^{(u)}$.

Double counting the set of incident point/line pairs in $\mathcal{S}$ yields (4). Double counting the set of pairs of lines (ℓ, ℓ') with ℓ and ℓ' lines of length j and ℓ and ℓ' intersecting yields (5). Double counting the set of pairs (ℓ, ℓ') with ℓ a line of length j_1 and ℓ' a line of length j_2 such that ℓ and ℓ' intersect yields (6). We remark that inequality may hold in the last two conditions because in a linear space a given pair of lines may or may not intersect. The condition (7) follows from counting points in the space. The condition (8) follows since the dual of the incidence relation between points $p \in \mathcal{V}$ such that $x_j \geq t$ and lines of size j is a pre-linear space with lines of length $x_j \geq t$. If necessary, we can shorten the lines sufficiently, so that we have a pre-linear space with $\sum_{\substack{u=1 \\ x_j^{(u)} \geq t}}^{U} y_u$ lines of length t. By adding in 2-lines for each pair of points that is not yet connected, we end up with a linear space on a_j points. Since $P(a_j, t)$ is an upper bound for the number of t-lines in any linear space on a_j points, the inequality must hold.

We conclude that $(y_1, \ldots, y_U)$ is a solution to the system of equalities and inequalities listed.

It remains to show that the decomposition that we obtain in this way is in fact the coarsest row-tactical refinement. To this end, let $\mathfrak{C} = (C_1, \ldots, C_U)$ be the partition (with possibly empty classes) of $\mathcal{V}$ that is obtained by collecting all points of $\mathcal{V}$ that have point type $\mathbf{x}^{(u)}$ in the class C_u. Moreover, let $\mathfrak{B}$ be the partition whose classes are the lines of any given length, i.e. $\mathfrak{B} = (\mathcal{L}_v, \mathcal{L}_{v-1}, \ldots, \mathcal{L}_2)$. Then $(\mathfrak{C}, \mathfrak{B})$ is a row-tactical decomposition of $\mathcal{S}$. Since $(\mathfrak{I}_{\mathcal{V}}, \mathfrak{B})$ is a column-tactical decomposition of $\mathcal{S}$, we must have that $\mathfrak{C} \preceq \mathfrak{V}'$ where $\mathfrak{V}'$ is the partition of points that is the coarsest row-tactical refinement of $(\mathfrak{I}_{\mathcal{V}}, \mathfrak{B})$ with respect to $\mathcal{S}$ (recall that $\mathfrak{I}_{\mathcal{V}}$ is the partition whose only class is the set $\mathcal{V}$). We claim that $\mathfrak{C} = \mathfrak{V}'$. To see this, assume the opposite. That means there are two (nonempty) classes C_s and C_t ($s \neq t$) such that $C_s \cup C_t$ is contained in the same class of $\mathfrak{V}'$. Now C_s and C_t are different classes in $\mathfrak{C}$ which means that their point type vectors $\mathbf{x}^{(s)}$ and $\mathbf{x}^{(t)}$ differ in at least one component. That is, there is a j ($2 \leq j \leq v$) such that $x_j^{(s)} \neq x_j^{(t)}$. Let p be a point in C_s and let q be a point in C_t. Then $|(p) \cap \mathcal{L}_j| = x_j^{(s)} \neq x_j^{(t)} = |(q) \cap \mathcal{L}_j|$. This means that the decomposition $\mathcal{V}'$ which we assume to have C_s and C_t together in one class is not row-tactical with respect to $\mathfrak{B}$.

This contradicts our assumption of $(\mathfrak{V}', \mathfrak{B})$ being the coarsest row-tactical refinement of $(\mathfrak{I}_{\mathcal{V}}, \mathfrak{B})$. Hence $\mathfrak{C} = \mathfrak{V}'$. $\square$

3.2 The General Row-Refinement

Let $(\mathfrak{V}, \mathfrak{A})$ be a row-tactical decomposition with structure constants $r_{i,j}$ for $i \in \mathbb{Z}_m$ and $j \in \mathbb{Z}_{n'}$. Let $(\mathfrak{V}, \mathfrak{B})$ be a column-tactical refinement with structure constants $k_{i,j}$ for $i \in \mathbb{Z}_m$ and $j \in \mathbb{Z}_n$. Our goal is to compute the row-tactical refinement $(\mathfrak{W}, \mathfrak{B})$ of the previous decomposition with structure constants $r'_{i,j}$ for $i \in \mathbb{Z}_{m'}$ and $j \in \mathbb{Z}_n$. We let $\mathfrak{A} = (A_1, \ldots, A_{n'})$, $\mathfrak{B} = (B_1, \ldots, B_n)$, $\mathfrak{V} = (V_1, \ldots, V_m)$ and $\mathfrak{W} = (W_1, \ldots, W_{m'})$. Also, we write a_i, b_i, v_i, w_i for the size of A_i, B_i, V_i, W_i, respectively.

For each $i \in \mathbb{Z}_m$, we introduce nonnegative integer variables $x_1^{(i)}, \ldots, x_n^{(i)}$. We then solve

$$\sum_{j=1}^{n} \max(k_{i,j} - 1, 0) x_j^{(i)} = v_i - 1, \quad \text{for all } i \in \mathbb{Z}_m, \tag{10}$$

and

$$\sum_{j=1}^{n} k_{i_2,j} x_j^{(i_1)} = v_{i_2} \quad \text{for all } i_1, i_2 \in \mathbb{Z}_m,\ i_1 \neq i_2, \tag{11}$$

and

$$\sum_{\substack{j=1,\ldots,n: \\ \mathfrak{A}(B_j)=A_J}} x_j^{(i)} = r_{i,J} \quad \text{for all } i \in \mathbb{Z}_m, \text{ and all } J \in \mathbb{Z}_{n'}, \tag{12}$$

with $x_j^{(i)} \leq b_j$ and specifically $x_j^{(i)} = 0$ if $k_{i,j} = 0$ for all $i \in \mathbb{Z}_m$ and all $j \in \mathbb{Z}_n$.

Let U_i be the number of solutions, and let $(x_1^{(i,u)}, \ldots, x_n^{(i,u)})$ for $u \in \mathbb{Z}_{U_i}$ be the u-th solution $(x_1^{(i)}, \ldots, x_n^{(i)})$. We introduce nonnegative integer variables $y_1^{(i)}, \ldots, y_{U_i}^{(i)}$ for $i \in \mathbb{Z}_m$. Then we solve

$$\sum_{i=1}^{m} \sum_{u=1}^{U_i} \binom{x_j^{(i,u)}}{2} y_u^{(i)} \leq \binom{b_j}{2} \quad \text{for all } j \in \mathbb{Z}_n, \tag{13}$$

and

$$\sum_{i=1}^{m} \sum_{u=1}^{U_i} x_{j_1}^{(i,u)} x_{j_2}^{(i,u)} y_u^{(i)} \leq b_{j_1} \cdot b_{j_2} \quad \text{for all } j_1, j_2 \in \mathbb{Z}_n, j_1 \neq j_2, \tag{14}$$

and

$$\sum_{u=1}^{U_i} x_j^{(i,u)} y_u^{(i)} = k_{i,j} b_j \quad \text{for all } i \in \mathbb{Z}_m, \text{ for all } j \in \mathbb{Z}_n, \tag{15}$$

and

$$\sum_{u=1}^{U_i} y_u^{(i)} = v_i, \quad \text{for all } i \in \mathbb{Z}_m, \tag{16}$$

and

$$\sum_{i=1}^{m} \sum_{\substack{u=1 \\ x_j^{(i,u)} \geq t}}^{U} y_u^{(i)} \leq P(b_j, t), \quad \text{for all } j \in \mathbb{Z}_n, \text{ for all } t \in \mathbb{Z}_{[3,b_j]}. \tag{17}$$

Let $y_u^{(i)}$ ($i \in \mathbb{Z}_m$, $u \in \mathbb{Z}_{U_i}$) be a nonnegative integer solution to these conditions. The row-scheme

$$\begin{array}{c|cccc} \rightarrow & b_1 & b_2 & \cdots & b_n \\ \hline y_1^{(1)} & x_1^{(1,1)} & x_2^{(1)} & \cdots & x_n^{(1)} \\ \vdots & \vdots & & & \vdots \\ y_{U_1}^{(1)} & x_1^{(1,U_1)} & x_2^{(1,U_1)} & \cdots & x_n^{(1,U_1)} \\ y_1^{(2)} & x_1^{(2,1)} & x_2^{(2,1)} & \cdots & x_n^{(2,1)} \\ \vdots & \vdots & & & \vdots \\ y_{U_{m'}}^{(m')} & x_1^{(m',U_{m'})} & x_2^{(m',U_{m'})} & \cdots & x_n^{(m',U_{m'})} \end{array} \tag{18}$$

is said to be obtained from the decompositions

$$\begin{array}{c|cccc} \rightarrow & a_1 & a_2 & \cdots & a_{n'} \\ \hline v_1 & r_{1,1} & r_{1,2} & \cdots & r_{1,n'} \\ \vdots & \vdots & & & \vdots \\ v_m & r_{m,1} & r_{m,2} & \cdots & r_{m,n'} \end{array} \quad \text{and} \quad \begin{array}{c|cccc} \downarrow & b_1 & b_2 & \cdots & b_n \\ \hline v_1 & k_{1,1} & k_{1,2} & \cdots & k_{1,n} \\ \vdots & \vdots & & & \vdots \\ v_m & k_{m,1} & k_{m,2} & \cdots & k_{m,n} \end{array}$$

by *(general) row-refinement.*

The following result is the analog of Lemma 4. We omit its proof.

Lemma 5 *Let $\mathcal{S}$ be a linear space. Let $(\mathfrak{V}, \mathfrak{A})$ be a row-tactical decomposition with structure constants $r_{i,j}$ for $i \in \mathbb{Z}_m$ and $j \in \mathbb{Z}_{n'}$. Let $(\mathfrak{V}, \mathfrak{B})$ be a column-tactical refinement with structure constants $k_{i,j}$ for $i \in \mathbb{Z}_m$ and $j \in \mathbb{Z}_n$. Then $\mathcal{S}$ satisfies exactly one of the row-tactical decompositions $(\mathfrak{W}, \mathfrak{B})$ with structure constants $r'_{i,j}$ that is obtained by general row-refinement. In fact, this row-tactical decomposition is the coarsest row-tactical decomposition of the column-tactical decomposition $(\mathfrak{V}, \mathfrak{B})$.*

3.3 The General Column-Refinement

Let $(\mathfrak{V}, \mathfrak{A})$ be a column-tactical decomposition with structure constants $k_{i,j}$ for $i \in \mathbb{Z}_{m'}$ and $j \in \mathbb{Z}_n$. Let $(\mathfrak{W}, \mathfrak{A})$ be a row-tactical refinement with structure constants $r_{i,j}$ for $i \in \mathbb{Z}_m$ and $j \in \mathbb{Z}_n$. Our goal is to compute the column-tactical refinement $(\mathfrak{W}, \mathfrak{B})$ of the previous decomposition with structure constants $k'_{i,j}$ for $i \in \mathbb{Z}_m$ and $j \in \mathbb{Z}_{n'}$. We let $\mathfrak{A} = (A_1, \ldots, A_n)$, $\mathfrak{B} = (B_1, \ldots, B_{n'})$, $\mathfrak{V} = (V_1, \ldots, V_{m'})$ and $\mathfrak{W} = (W_1, \ldots, W_m)$. Also, we write a_i, b_i, v_i, w_i for the size of A_i, B_i, V_i, W_i, respectively.

For each $j \in \mathbb{Z}_n$, we introduce nonnegative integer variables $x_1^{(j)}, \ldots, x_m^{(j)}$. We then solve

$$\sum_{i=1}^{m} \max(r_{i,j} - 1, 0) x_i^{(j)} \le a_j - 1, \quad \text{for all } j \in \mathbb{Z}_n, \tag{19}$$

and

$$\sum_{i=1}^{m} r_{i,j_2} x_i^{(j_1)} \le a_{j_2}, \quad \text{for all } j_1, j_2 \in \mathbb{Z}_n,\ j_1 \ne j_2 \tag{20}$$

and

$$\sum_{\substack{i=1 \\ \mathfrak{V}(W_i)=V_I}}^{m} x_i^{(j)} = k_{I,j}, \quad \text{for all } I \in \mathbb{Z}_{m'}, \tag{21}$$

subject to $x_i^{(j)} \le w_i$ and specifically $x_i^{(j)} = 0$ if $r_{i,j} = 0$ for $i \in \mathbb{Z}_m$, $j \in \mathbb{Z}_n$.

Let U_j be the number of solutions, and let $(x_1^{(j,u)}, \ldots, x_m^{(j,u)})$ for $u \in \mathbb{Z}_{U_j}$ be the u-th solution $(x_1^{(j)}, \ldots, x_m^{(j)})$. We introduce nonnegative integer variables $y_1^{(j)}, \ldots, y_{U_j}^{(j)}$ for $j \in \mathbb{Z}_n$. Then we solve

$$\sum_{j=1}^{n} \sum_{u=1}^{U_j} \binom{x_i^{(j,u)}}{2} y_u^{(j)} = \binom{w_i}{2} \quad \text{for } i \in \mathbb{Z}_m, \tag{22}$$

and

$$\sum_{j=1}^{n}\sum_{u=1}^{U_j} x_{i_1}^{(j,u)} x_{i_2}^{(j,u)} y_u^{(j)} = w_{i_1} \cdot w_{i_2} \quad \text{for } i_1, i_2 \in \mathbb{Z}_m,\ i_1 \neq i_2, \tag{23}$$

and

$$\sum_{u=1}^{U_j} x_i^{(j,u)} y_u^{(j)} = r_{i,j} w_i \quad \text{for } i \in \mathbb{Z}_m, \text{for } j \in \mathbb{Z}_n, \tag{24}$$

and

$$\sum_{u=1}^{U_j} y_u^{(j)} = a_j \quad \text{for } j \in \mathbb{Z}_n, \tag{25}$$

and

$$\sum_{j=1}^{n} \sum_{\substack{u=1 \\ x_i^{(j,u)} \geq t}}^{U} y_u^{(j)} \leq P(w_i, t), \quad \text{for all } i \in \mathbb{Z}_m,\ \text{for all } t \in \mathbb{Z}_{[3,w_i]}. \tag{26}$$

Let $y_u^{(j)}$ ($j \in \mathbb{Z}_m$, $u \in \mathbb{Z}_{U_j}$) be a nonnegative integer solution to these systems of equations and inequalities. The column-scheme

$$\begin{array}{c|cccccc} \downarrow & y_1^{(1)} & \cdots & y_{U_1}^{(1)} & y_1^{(2)} & \ldots & y_{U_n}^{(n)} \\ \hline w_1 & x_1^{(1,1)} & \cdots & x_1^{(1,U_1)} & x_1^{(2,1)} & & x_1^{(n,U_n)} \\ \vdots & \vdots & & & & & \vdots \\ w_m & x_m^{(1,1)} & \cdots & x_m^{(1,U_1)} & x_m^{(2,1)} & & x_m^{(n,U_n)} \end{array} \tag{27}$$

is said to be obtained from the decompositions

$$\begin{array}{c|cccc} \downarrow & a_1 & a_2 & \cdots & a_n \\ \hline v_1 & k_{1,1} & k_{1,2} & \cdots & k_{1,n} \\ \vdots & \vdots & & & \vdots \\ v_{m'} & k_{m',1} & k_{m',2} & \cdots & k_{m',n} \end{array} \quad \text{and} \quad \begin{array}{c|cccc} \rightarrow & b_1 & b_2 & \cdots & b_n \\ \hline w_1 & r_{1,1} & r_{1,2} & \cdots & r_{1,n} \\ \vdots & \vdots & & & \vdots \\ w_m & r_{m,1} & r_{m,2} & \cdots & r_{m,n} \end{array}$$

by *(general) column-refinement.*

The following result is the analog of Lemma 4 for column refinements. We omit its proof.

Lemma 6 *Let $\mathcal{S}$ be a linear space. Let $(\mathfrak{V}, \mathfrak{A})$ be a column-tactical decomposition with structure constants $k_{i,j}$ for $i \in \mathbb{Z}_{m'}$ and $j \in \mathbb{Z}_n$. Let $(\mathfrak{W}, \mathfrak{A})$ be a row-tactical refinement with structure constants $r_{i,j}$ for $i \in \mathbb{Z}_m$ and $j \in \mathbb{Z}_n$. Then $\mathcal{S}$ satisfies exactly one of the column-tactical decompositions $(\mathfrak{W}, \mathfrak{B})$ with structure constants $k'_{i,j}$ that is obtained by general column-refinement. In fact, this column-tactical decomposition is the coarsest column-tactical decomposition of the row-tactical decomposition $(\mathfrak{W}, \mathfrak{A})$.*

4 Case 3

Let $\mathcal{S} = (\mathcal{V}, \mathcal{L})$ be a linear space on 30 points with line type 3. We assume that $\mathcal{S}$ admits the column-tactical decomposition

Case3

		$\mathcal{L}_7$	$\mathcal{L}_5$	$\mathcal{L}_4$
	$\downarrow$	3	15	37
$\mathcal{V}$	30	7	5	4

(28)

The point types (x_1, x_2, x_3) are the solutions of the equation $6x_1 + 4x_2 + 3x_3 = 29$ subject to $x_1 \leq 3, x_2 \leq 15$ and $x_3 \leq 37$. The 6 solutions for (x_1, x_2, x_3) are $(3, 2, 1)$, $(2, 2, 3)$, $(1, 5, 1)$, $(1, 2, 5)$, $(0, 5, 3)$, $(0, 2, 7)$. The reduced system to compute the distributions is listed in Tab. 2, together with the 10 solutions. Only Case3.2 and Case3.4 allow column-tactical refinements (i.e., the other Cases are eliminated). Case3.2 has 38 refinements Case3.2.i ($i = 1, \ldots, 38$) and Case3.4 has 457 refinements Case3.4.i ($i = 1, \ldots, 457$). We display the row-tactical decomposition Case3.2 and Case3.4:

Case3.2

$\rightarrow$	3	15	37
1	3	2	1
18	1	2	5
5	0	5	3
6	0	2	7

Case3.4

$\rightarrow$	3	15	37
3	2	2	3
15	1	2	5
5	0	5	3
7	0	2	7

(29)

A computer search was performed which showed (in around 5 days CPU time) that the two decompositions Case3.2 and Case3.4 are not realizable. Thus there is no linear space in this case.

y_1	y_2	y_3	y_4	y_5	y_6		
3	2	1	1	0	0	$=21$	F_1
2	2	5	2	5	2	$=75$	F_2
3	1	0	0	0	0	≤ 3	J_1
6	4	5	2	0	0	≤ 45	$J_{1,2}$
solutions:							
1	0	1	17	4	7	Case3.1	
1	0	0	18	5	6	Case3.2	
0	3	1	14	4	8	Case3.3	
0	3	0	15	5	7	Case3.4	
0	2	1	16	4	7	Case3.5	
0	2	0	17	5	6	Case3.6	
0	1	1	18	4	6	Case3.7	
0	1	0	19	5	5	Case3.8	
0	0	1	20	4	5	Case3.9	
0	0	0	21	5	4	Case3.10	

Table 2: Computing Point Distributions in Case 3

5 Case 4

Let $\mathcal{S} = (\mathcal{V}, \mathcal{L})$ be a linear space on 30 points with line type 4. We assume that $\mathcal{S}$ admits the column-tactical decomposition

$$\begin{array}{c} \text{Case4} \\ \begin{array}{cc|ccc} & & \mathcal{L}_7 & \mathcal{L}_5 & \mathcal{L}_4 \\ & \downarrow & 1 & 27 & 24 \\ \hline \mathcal{V} & 30 & 7 & 5 & 4 \end{array} \end{array} \tag{30}$$

The point types (x_1, x_2, x_3) are the solutions of the equation $6x_1+4x_2+3x_3 = 29$ subject to $x_1 \leq 1, x_2 \leq 27$ and $x_3 \leq 24$. The 4 solutions (x_1, x_2, x_3) are $(1, 5, 1)$, $(1, 2, 5)$, $(0, 5, 3)$, $(0, 2, 7)$. The reduced system to compute the distributions is listed in Tab. 3, together with the 2 solutions. The resulting

y_1	y_2	y_3	y_4		
1	1	0	0	$=7$	F_1
5	2	5	2	$=135$	F_2
5	2	0	0	≤ 27	$J_{1,2}$
1	5	0	0	≤ 24	$J_{1,3}$
Solutions:					
4	3	21	2	Case4.1	
3	4	22	1	Case4.2	

Table 3: Computing Point Distributions in Case 4

two possible row-tactical refinements are

Case4.1

$\rightarrow$	1	27	24
4	1	5	1
3	1	2	5
21	0	5	3
2	0	2	7

Case4.2

$\rightarrow$	1	27	24
3	1	5	1
4	1	2	5
22	0	5	3
1	0	2	7

Consider first Case4.1. There are 26 5-lines and 19 4-lines that intersect the 7-line. This means that we have the column tactical decomposition

	$\mathcal{L}_7$	$\mathcal{L}_{5,2}$	$\mathcal{L}_{5,1}$	M	L
$\downarrow$	1	26	1	19	5
7	7	1	0	1	0
23	0	4	5	3	4

(31)

We compute refined point types using the conditions displayed in Tab. 4. The resulting point types are listed in Tab. 5. The distribution of point types is computed using the conditions shown in Tab. 6. There are exactly 2 solutions, also shown in the table. Thus we find the following two row-tactical

$x_1^{(1)}$	$s_1^{(1)}$	$x_2^{(1)}$	$x_3^{(1)}$	$s_2^{(1)}$	$x_4^{(1)}$	$x_5^{(1)}$	$s_3^{(1)}$	
1	1	26	1	27	19	5	24	bound
1	1	1	1	1	1	1	1	= 52
6	0	0	0	0	0	0	0	= 6
0	0	4	5	0	3	4	0	= 23
1	1	0	0	0	0	0	0	= 1
0	0	1	1	1	0	0	0	= 27
0	0	0	0	0	1	1	1	= 24
$x_1^{(2)}$	$s_1^{(2)}$	$x_2^{(2)}$	$x_3^{(2)}$	$s_2^{(2)}$	$x_4^{(2)}$	$x_5^{(2)}$	$s_3^{(2)}$	
1	1	26	1	27	19	5	24	bound
1	1	1	1	1	1	1	1	= 52
7	0	1	0	0	1	0	0	= 7
0	0	3	4	0	2	3	0	= 22
1	1	0	0	0	0	0	0	= 1
0	0	1	1	1	0	0	0	= 27
0	0	0	0	0	1	1	1	= 24

Table 4: Computing Point Types in Case4.1

i	point type	i	point type	i	point type
1	$(1,5,0,1,0)$	1	$(1,1,2,3,0)$	1	$(1,0,0,5,2)$
1	$(1,4,0,1,1)$	1	$(1,1,1,2,2)$	1	$(1,0,0,1,5)$
1	$(1,3,1,2,0)$	1	$(1,1,0,5,1)$	2	$(1,0,4,0,2)$
1	$(1,3,0,1,2)$	1	$(1,1,0,1,4)$	2	$(1,0,1,0,6)$
1	$(1,2,3,0,0)$	1	$(1,0,4,1,0)$	2	$(0,5,0,2,1)$
1	$(1,2,1,2,1)$	1	$(1,0,3,0,2)$	2	$(0,4,1,3,0)$
1	$(1,2,0,5,0)$	1	$(1,0,2,3,1)$	2	$(0,2,0,5,2)$
1	$(1,2,0,1,3)$	1	$(1,0,1,6,0)$	2	$(0,1,1,6,1)$
1	$(1,1,3,0,1)$	1	$(1,0,1,2,3)$	2	$(0,0,2,7,0)$

Table 5: Point Types in Case4.1

0	0	0	1	0	0	3	1	0	6	0	3	1	10	0	0	1	0	≤ 10	J_5
0	0	2	0	2	0	0	2	0	0	6	2	0	0	0	3	0	6	≤ 19	$J_{3,4}$
5	4	3	3	2	2	2	1	1	1	0	0	0	0	0	0	0	0	$= 26$	$F_{2,1}$
1	1	2	1	2	5	1	2	5	1	6	2	5	1	0	0	0	0	$= 19$	$F_{4,1}$
0	0	0	0	0	0	0	0	0	0	0	0	0	0	5	4	2	1	$= 104$	$F_{2,2}$
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	1	$= 5$	$F_{3,2}$
Solutions:																			
4	0	0	0	0	3	0	0	0	0	0	0	0	0	17	4	1	1	Case4.1.1	
4	0	0	0	0	3	0	0	0	0	0	0	0	0	16	5	2	0	Case4.1.2	

Table 6: Computing Point Distributions in Case4.1

refinements:

Case4.1.1

$\rightarrow$	1	26	1	19	5
4	1	5	0	1	0
3	1	2	0	5	0
17	0	5	0	2	1
4	0	4	1	3	0
1	0	2	0	5	2
1	0	1	1	6	1

Case4.1.2

$\rightarrow$	1	26	1	19	5
4	1	5	0	1	0
3	1	2	0	5	0
16	0	5	0	2	1
5	0	4	1	3	0
2	0	2	0	5	2

Starting with Case4.2, we obtain the following two decompositions (details omitted):

Case4.2.1

$\rightarrow$	1	23	4	23	1
3	1	5	0	1	0
4	1	2	0	5	0
4	0	5	0	2	1
18	0	4	1	3	0
1	0	0	2	7	0

Case4.2.2

$\rightarrow$	1	23	4	23	1
3	1	5	0	1	0
4	1	2	0	5	0
3	0	5	0	2	1
19	0	4	1	3	0
1	0	1	1	6	1

A computer search was performed which showed (in around 2 days CPU time) that the decompositions Case4.1.1, Case 4.1.2, Case4.2.1 and Case4.2.2 are not realizable. Thus there is no linear space in this case.

6 Case 5

Let $\mathcal{S} = (\mathcal{V}, \mathcal{L})$ be a linear space on 30 points with line type 5. We assume that $\mathcal{S}$ admits the column-tactical decomposition

Case5

		$\mathcal{L}_7$	$\mathcal{L}_5$	$\mathcal{L}_4$
	$\downarrow$	1	24	29
$\mathcal{V}$	30	7	5	4

(32)

The two row-tactical refinements are

$\rightarrow$	1	24	29
3	1	5	1
4	1	2	5
17	0	5	3
6	0	2	7

$\rightarrow$	1	24	29
2	1	5	1
5	1	2	5
18	0	5	3
5	0	2	7

Let a and b be the number of 5-lines and 4-lines (respectively) intersecting the 7-line. Then $(a, b) = (23, 23)$ in the first case and $(a, b) = (20, 27)$ in the second case. Hence we have the two column-tactical decompositions

Case5.1

$\downarrow$	1	23	1	23	6
7	7	1	0	1	0
23	0	4	5	3	4

Case5.2

$\downarrow$	1	20	4	27	2
7	7	1	0	1	0
23	0	4	5	3	4

Proceeding in a way that is completely analogous to what was done in Section 5, we find 6 possible row-tactical refinements (two are refinements of Case5.1, four are refinements of Case5.2). These 6 decompositions are

Case5.1.1

$\rightarrow$	1	23	1	23	6
3	1	5	0	1	0
4	1	2	0	5	0
13	0	5	0	2	1
4	0	4	1	3	0
5	0	2	0	5	2
1	0	1	1	6	1

Case5.1.2

$\rightarrow$	1	23	1	23	6
3	1	5	0	1	0
4	1	2	0	5	0
12	0	5	0	2	1
5	0	4	1	3	0
6	0	2	0	5	2

Case5.2.1

→	1	20	4	27	2
2	1	5	0	1	0
5	1	2	0	5	0
4	0	5	0	2	1
14	0	4	1	3	0
4	0	1	1	6	1
1	0	0	2	7	0

Case5.2.2

→	1	20	4	27	2
2	1	5	0	1	0
5	1	2	0	5	0
3	0	5	0	2	1
15	0	4	1	3	0
1	0	2	0	5	2
3	0	1	1	6	1
1	0	0	2	7	0

Case5.2.3

→	1	20	4	27	2
2	1	5	0	1	0
5	1	2	0	5	0
3	0	5	0	2	1
15	0	4	1	3	0
5	0	1	1	6	1

Case5.2.4

→	1	20	4	27	2
2	1	5	0	1	0
5	1	2	0	5	0
2	0	5	0	2	1
16	0	4	1	3	0
1	0	2	0	5	2
4	0	1	1	6	1

A computer search was performed which showed (in around 2 days CPU time) that the decompositions Case5.1.i ($i = 1, 2$) and Case 5.2.i ($i = 1, \dots, 4$) are not realizable. Thus there is no linear space in this case.

7 Case 6

In the following, assume that $\mathcal{S} = (\mathcal{V}, \mathcal{L})$ is a linear space on 30 points with line type 6, i.e., with one 7-line, 15 five-lines and 44 four-lines. Throughout, let $\mathcal{L}_7, \mathcal{L}_5, \mathcal{L}_4$ denote the set of 7-lines, 5-lines, 4-lines, respectively. Clearly, we assume that $\mathcal{S}$ admits the column-tactical decomposition

Case6

		$\mathcal{L}_7$	$\mathcal{L}_5$	$\mathcal{L}_4$
	↓	1	15	44
$\mathcal{V}$	30	7	5	4

(33)

Lemma 7 *Let C denote the set of points that lie on the 7-line. Each point in C lies on two 5-lines and on five 4-lines. The points off the 7-line fall into*

y_1	y_2	y_3	y_4		
1	1	0	0	$= 7$	F_1
5	2	5	2	$= 75$	F_2
5	2	0	0	≤ 15	$J_{1,2}$

Table 7: Case 6, Computing Refined Point Types

two disjoint classes D and E, with $|D| = 5$ and $|E| = 18$. Each point in D lies on exactly five 5-lines and on exactly three 4-lines. Each point in E lies on exactly two 5-lines and on exactly seven 4-lines. Equivalently, $\mathcal{S}$ admits the point tactical refinement of (33)

$$\begin{array}{cc|ccc} & & \mathcal{L}_7 & \mathcal{L}_5 & \mathcal{L}_4 \\ & \rightarrow & 1 & 15 & 44 \\ \hline C & 7 & 1 & 2 & 5 \\ D & 5 & 0 & 5 & 3 \\ E & 18 & 0 & 2 & 7 \end{array} \tag{34}$$

Proof. The point types are determined as the solutions to the equation $6x_1 + 4x_2 + 3x_3 = 29$ subject to $x_1 \leq 1, x_2 \leq 15$ and $x_3 \leq 44$. The four solutions (x_1, x_2, x_3) are $(1,5,1)$,$(1,2,5)$,$(0,5,3)$,$(0,2,7)$.Let y_i be the number of points of the i-th type. The y_i satisfy the system of equalities and inequalities displayed in Tab. 7. The only solution is $(y_1, y_2, y_3, y_4) = (0, 7, 5, 18)$. Let C, D, E be the set of points of type $2, 3, 4$, respectively. This yields the required row-tactical decomposition. □

Lemma 8 *The linear space $\mathcal{S}$ admits exactly one of the following two column tactical decompositions refining (34), which we call Case 6A and Case 6B, respectively.*

Case 6A

$$\begin{array}{cc|cccccccc} & & \mathcal{L}_7 & \mathcal{L}_{5,2a} & \mathcal{L}_{5,2b} & \mathcal{L}_{5,1} & M_1 & M_2 & L_1 & L_2 \\ & \downarrow & 1 & 10 & 4 & 1 & 11 & 24 & 4 & 5 \\ \hline C & 7 & 7 & 1 & 1 & 0 & 1 & 1 & 0 & 0 \\ D & 5 & 0 & 2 & 1 & 1 & 1 & 0 & 1 & 0 \\ E & 18 & 0 & 2 & 3 & 4 & 2 & 3 & 3 & 4 \end{array} \tag{35}$$

Case $6B$

		$\mathcal{L}_7$	$\mathcal{L}_{5,2a}$	$\mathcal{L}_{5,2b}$	$\mathcal{L}_{5,1}$	M_1	M_2	L_1	L_2
	$\downarrow$	1	9	5	1	12	23	3	6
C	7	7	1	1	0	1	1	0	0
D	5	0	2	1	2	1	0	1	0
E	18	0	2	3	3	2	3	3	4

(36)

In particular, there is exactly one 5*-line disjoint from the* 7*-line, and there are exactly* 9 *four-lines disjoint from the* 7*-line.*

Proof. We compute the refined line types of the row-tactical decomposition from Lemma 7. The points in C, D, E, are said to be of type $1, 2, 3$, respectively. A line is said to be of type $1, 2, 3$, if it has length 7, 5 or 4, respectively. Let $x_i^{(j)}$ be the number of points of type i ($i = 1, 2, 3$) that are incident with a line of type j. The 7-line comprises all C-points, and hence $(x_1^{(1)}, x_2^{(1)}, x_3^{(1)}) = (7, 0, 0) = \mathbf{x}_1^{(1)}$. The refined types $(x_1^{(2)}, x_2^{(2)}, x_3^{(2)})$ of 5-lines are the solutions to the system

$x_1^{(2)}$	$x_2^{(2)}$	$x_3^{(2)}$	
1	1	1	$= 5$
1	4	1	≤ 14
1	0	0	≤ 1
5	3	7	≤ 44

subject to the conditions $x_1^{(2)} \leq 1$, $x_2^{(2)} \leq 5$, and $x_3^{(2)} \leq 18$. This yields the possibilities

$$(x_1^{(2)}, x_2^{(2)}, x_3^{(2)}) = \begin{cases} (1,3,1) = \mathbf{x}_1^{(2)} \\ (1,2,2) = \mathbf{x}_2^{(2)} \\ (1,1,3) = \mathbf{x}_3^{(2)} \\ (1,0,4) = \mathbf{x}_4^{(2)} \\ (0,3,2) = \mathbf{x}_5^{(2)} \\ (0,2,3) = \mathbf{x}_6^{(2)} \\ (0,1,4) = \mathbf{x}_7^{(2)} \\ (0,0,5) = \mathbf{x}_8^{(2)} \end{cases}$$

$y_1^{(2)}$	$y_2^{(2)}$	$y_3^{(2)}$	$y_4^{(2)}$	$y_5^{(2)}$	$y_6^{(2)}$	$y_7^{(2)}$	$y_8^{(2)}$	$y_1^{(3)}$	$y_2^{(3)}$	$y_3^{(3)}$	$y_4^{(3)}$	$y_5^{(3)}$	$y_6^{(3)}$		
3	1	0	0	3	1	0	0	1	0	0	1	0	0	$=10$	J_2
0	1	3	6	1	3	6	10	0	1	3	1	3	6	$=153$	J_3
1	2	3	4	0	0	0	0	1	2	3	0	0	0	$=126$	$J_{1,3}$
3	4	3	0	6	6	4	0	2	2	0	4	3	0	$=90$	$J_{2,3}$
1	1	1	1	0	0	0	0	0	0	0	0	0	0	$=14$	$F_{2,1}$
3	2	1	0	3	2	1	0	0	0	0	0	0	0	$=25$	$F_{2,2}$
Solutions:															
0	10	4	0	0	0	1	0	0	11	24	0	4	5	Case6A	
0	9	5	0	0	1	0	0	0	12	23	0	3	6	Case6B	

Table 8: Case 6, Computing Refined Line Types

The refined types $(x_1^{(3)}, x_2^{(3)}, x_3^{(3)})$ of 4-lines are the solutions to the system

$x_1^{(3)}$	$x_2^{(3)}$	$x_3^{(3)}$	
1	1	1	$=4$
4	2	6	≤ 43
1	0	0	≤ 1
2	5	2	≤ 15

subject to the conditions $x_1^{(3)} \leq 1$, $x_2^{(3)} \leq 5$, and $x_3^{(3)} \leq 18$. This yields the possibilities

$$(x_1^{(3)}, x_2^{(3)}, x_3^{(3)}) = \begin{cases} (1,2,1) = \mathbf{x}_1^{(3)} \\ (1,1,2) = \mathbf{x}_2^{(3)} \\ (1,0,3) = \mathbf{x}_3^{(3)} \\ (0,2,2) = \mathbf{x}_4^{(3)} \\ (0,1,3) = \mathbf{x}_5^{(3)} \\ (0,0,4) = \mathbf{x}_6^{(3)} \end{cases}$$

Let $y_{j_i}^{(j)}$ be the number of lines of type j that have refined line type $\mathbf{x}_{j_i}^{(j)}$. We know that $y_1^{(1)} = 1$. The refined line type distributions are computed using the conditions displayed in Tab. 8 subject to the conditions $y_i^{(2)} \leq 15$ and $y_i^{(3)} \leq 44$. There are exactly two solutions, also shown in Tab. 8. Thus, we arrive at the two decomposition schemes Case 6A and Case 6B. $\square$

Let ℓ_5 denote the 5-line disjoint from the 7-line, so that $\mathcal{L}_{5,1} = \{\ell_5\}$ and $\mathcal{L}_{5,2} = \mathcal{L}_{5,2a} \cup \mathcal{L}_{5,2b} = \mathcal{L}_5 \setminus \mathcal{L}_{5,1}$. Let L denote the nine 4-lines disjoint from the 7-line, so that $L = L_1 \cup L_2$. Let M be the 35 other 4-lines, so that $M = M_1 \cup M_2$. Let $D_5 = \ell_5 \cap D$ and let $E_5 = \ell_5 \cap E$. Moreover, let $D_0 = D \setminus D_5$ and let $E_0 = E \setminus E_5$. We observe that the size of D_5 is either one or two. That is, the 5-line that is disjoint from the 7-line contains either one or exactly two points of D. Accordingly, the size of E_5 is either 4 or 3. Put $D_0 = D \setminus D_5$ and $E_0 = E \setminus E_5$.

Lemma 9 *The linear space $\mathcal{S}$ admits the following row-tactical refinement of (34):*

	$\rightarrow$	$\mathcal{L}_7$ 1	$\mathcal{L}_{5,2}$ 14	$\mathcal{L}_{5,1}$ 1	M 35	L 9
C	7	1	2	0	5	0
D_0	4	0	5	0	2	1
D_5	1	0	4	1	3	0
E_0	14	0	2	0	5	2
E_5	4	0	1	1	6	1

(37)

Proof. We know from Lemma 8 that ℓ_5 is the unique 5-line disjoint from the 7-line and that there is a set L of nine 4-lines that are also disjoint from the 7-line. That is, we have a column tactical decomposition

	$\downarrow$	$\mathcal{L}_7$ 1	$\mathcal{L}_{5,2}$ 14	$\mathcal{L}_{5,1}$ 1	M 35	L 9
C	7	7	1	0	1	0
$D \cup E$	23	0	4	5	3	4

(38)

We claim that this decomposition has the row-tactical refinement

	$\rightarrow$	$\mathcal{L}_7$ 1	$\mathcal{L}_{5,2}$ 14	$\mathcal{L}_{5,1}$ 1	M 35	L 9
C	7	1	2	0	5	0
D_0	4	0	5	0	$c_{1,1}$	$c_{1,2}$
D_5	1	0	4	1	$c_{2,1}$	$c_{2,2}$
E_0	14	0	2	0	$c_{3,1}$	$c_{3,2}$
E_5	4	0	1	1	$c_{4,1}$	$c_{4,2}$

To this end, we recall that $c_{1,1} + c_{1,2} = c_{2,1} + c_{2,2} = 3$ and $c_{3,1} + c_{3,2} = c_{4,1} + c_{4,2} = 7$ from (34). Fix $p \in D_0$. Since every point in $D \cup E \setminus \{p\}$ is joined to p, we find that $5 \times (4-1) + c_{1,1} \times (3-1) + c_{1,2} \times (4-1) = 22$, i.e., $2c_{1,1} + 3c_{1,2} = 7$. Thus $c_{1,1} = 2$ and $c_{1,2} = 1$. Similarly, for $p \in D_5$ we obtain $4 \times (4-1) + 1 \times (5-1) + c_{2,1} \times (3-1) + c_{2,2} \times (4-1) = 22$, i.e., $2c_{2,1} + 3c_{2,2} = 6$. Since $c_{2,1} + c_{2,2} = 3$, this forces $c_{2,1} = 3$ and $c_{2,2} = 0$. The remaining cases are handled in a similar way and hence omitted. □

Lemma 10 *The linear spaces of Case* $6A$ *admit a row-tactical decomposition of type Case* $6A.\nu$ *with* $\nu = 1, \ldots, 82$*. The linear spaces of Case* $6B$ *admit a row-tactical decomposition of type Case* $6B.\mu$ *with* $\mu = 1, \ldots, 89$*. Here,*

Cases $6A.\nu$

		$\mathcal{L}_7$	$\mathcal{L}_{5,2a}$	$\mathcal{L}_{5,2b}$	$\mathcal{L}_{5,1}$	M	L
	$\rightarrow$	1	10	4	1	35	9
C_1	$y_1^{(1)}(\nu)$	1	2	0	0	5	0
C_2	$y_2^{(1)}(\nu)$	1	1	1	0	5	0
C_3	$y_4^{(1)}(\nu)$	1	0	2	0	5	0
D_0	$y_1^{(2)}(\nu)$	0	4	1	0	2	1
D_5	$y_2^{(2)}(\nu)$	0	4	0	1	3	0
$E_{0,1}$	$y_1^{(3)}(\nu)$	0	2	0	0	5	2
$E_{0,2}$	$y_2^{(3)}(\nu)$	0	1	1	0	5	2
$E_{0,3}$	$y_4^{(3)}(\nu)$	0	0	2	0	5	2
$E_{5,1}$	$y_3^{(3)}(\nu)$	0	1	0	1	6	1
$E_{5,2}$	$y_5^{(3)}(\nu)$	0	0	1	1	6	1

(39)

with $y_j^{(i)}(\nu)$ *a solution to the equations in Tab. 11 and*

Cases $6B.\mu$

		$\mathcal{L}_7$	$\mathcal{L}_{5,2a}$	$\mathcal{L}_{5,2b}$	$\mathcal{L}_{5,1}$	M	L
	$\rightarrow$	1	9	5	1	35	9
C_1	$y_1^{(1)}(\mu)$	1	2	0	0	5	0
C_2	$y_2^{(1)}(\mu)$	1	1	1	0	5	0
C_3	$y_4^{(1)}(\mu)$	1	0	2	0	5	0
D_0	$y_1^{(2)}(\mu)$	0	4	1	0	2	1
D_5	$y_3^{(2)}(\mu)$	0	3	1	1	3	0
$E_{0,1}$	$y_1^{(3)}(\mu)$	0	2	0	0	5	2
$E_{0,2}$	$y_2^{(3)}(\mu)$	0	1	1	0	5	2
$E_{0,3}$	$y_4^{(3)}(\mu)$	0	0	2	0	5	2
$E_{5,1}$	$y_3^{(3)}(\mu)$	0	1	0	1	6	1
$E_{5,2}$	$y_5^{(3)}(\mu)$	0	0	1	1	6	1

(40)

with $y_j^{(i)}(\mu)$ *a solution to the equations in Tab. 12. In all cases,* $C = C_1 \cup C_2 \cup C_3$*,* $E_0 = E_{0,1} \cup E_{0,2} \cup E_{0,3}$ *and* $E_5 = E_{5,1} \cup E_{5,2}$*.*

Proof. Using the conditions displayed in Tab. 9, we compute the refined point types for each of the three point classes C, D and E with respect to the first four column classes. The solutions are listed in Tab. 10.

The next step is to compute the partial row-tactical refinements. Tables 11 and 12 show the systems for Case $6A$ and Case $6B$, respectively. There are 82 solutions for Case $6A$ and 89 solutions for Case $6B$. Restricting to the point types in Tab. 10 that occur in these solutions leads to the two row-tactical decomposition schemes displayed. □

A computer search was performed which showed (in around 5 years CPU time) that the decompositions Case 6A.ν ($\nu = 1, \ldots, 82$) and Case 6B.μ ($\mu = 1, \ldots, 89$) are not realizable. Thus there is no linear space in this case.

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Case $6A$					
i	$x_1^{(i)}$	$x_2^{(i)}$	$x_3^{(i)}$	$x_4^{(i)}$	
1	1	2	2	1	bound
1	1	0	0	0	$= 1$
1	0	1	1	1	$= 2$
1	6	0	0	0	≤ 6
1	0	2	1	1	≤ 5
1	0	2	3	4	≤ 18
2	0	5	4	1	bound
2	1	0	0	0	$= 0$
2	0	1	1	1	$= 5$
2	7	1	1	0	≤ 7
2	0	1	0	0	≤ 4
2	0	2	3	4	≤ 18
3	0	2	2	1	bound
3	1	0	0	0	$= 0$
3	0	1	1	1	$= 2$
3	7	1	1	0	≤ 7
3	0	2	1	1	≤ 5
3	0	1	2	3	≤ 17

Case $6B$					
i	$x_1^{(i)}$	$x_2^{(i)}$	$x_3^{(i)}$	$x_4^{(i)}$	
1	1	2	2	1	bound
1	1	0	0	0	$= 1$
1	0	1	1	1	$= 2$
1	6	0	0	0	≤ 6
1	0	2	1	2	≤ 5
1	0	2	3	3	≤ 18
2	0	5	5	1	bound
2	7	1	1	0	≤ 7
2	0	1	0	1	≤ 4
2	0	2	3	3	≤ 18
2	1	0	0	0	$= 0$
2	0	1	1	1	$= 5$
3	0	2	2	1	bound
3	7	1	1	0	≤ 7
3	0	2	1	2	≤ 5
3	0	1	2	2	≤ 17
3	1	0	0	0	$= 0$
3	0	1	1	1	$= 2$

Table 9: Case 6, Conditions For Partially Refined Point Types

Case $6A$				Case $6B$			
i	j	$\mathbf{x}_j^{(i)}$	occurs	i	j	$\mathbf{x}_j^{(i)}$	occurs
1	1	$(1,2,0,0)$	yes	1	1	$(1,2,0,0)$	yes
1	2	$(1,1,1,0)$	yes	1	2	$(1,1,1,0)$	yes
1	3	$(1,1,0,1)$	no	1	3	$(1,1,0,1)$	no
1	4	$(1,0,2,0)$	yes	1	4	$(1,0,2,0)$	yes
1	5	$(1,0,1,1)$	no	1	5	$(1,0,1,1)$	no
2	1	$(0,4,1,0)$	yes	2	1	$(0,4,1,0)$	yes
2	2	$(0,4,0,1)$	yes	2	2	$(0,3,2,0)$	no
2	3	$(0,3,2,0)$	no	2	3	$(0,3,1,1)$	yes
2	4	$(0,3,1,1)$	no	2	4	$(0,2,3,0)$	no
2	5	$(0,2,3,0)$	no	2	5	$(0,2,2,1)$	no
2	6	$(0,2,2,1)$	no	2	6	$(0,1,4,0)$	no
2	7	$(0,1,4,0)$	no	2	7	$(0,1,3,1)$	no
2	8	$(0,1,3,1)$	no	2	8	$(0,0,5,0)$	no
2	9	$(0,0,4,1)$	no	2	9	$(0,0,4,1)$	no
3	1	$(0,2,0,0)$	yes	3	1	$(0,2,0,0)$	yes
3	2	$(0,1,1,0)$	yes	3	2	$(0,1,1,0)$	yes
3	3	$(0,1,0,1)$	yes	3	3	$(0,1,0,1)$	yes
3	4	$(0,0,2,0)$	yes	3	4	$(0,0,2,0)$	yes
3	5	$(0,0,1,1)$	yes	3	5	$(0,0,1,1)$	yes

Table 10: Case 6, Partially Refined Point Types

Case 6A																				
0	0	0	1	0	0	0	1	0	3	1	6	3	6	0	0	0	1	0	≤ 6	J_3
0	0	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	≤ 1	$J_{1,4}$
0	0	1	0	0	0	4	0	3	0	2	0	1	0	0	0	1	0	0	≤ 10	$J_{2,4}$
2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	$= 10$	$F_{2,1}$
0	1	0	2	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	$= 4$	$F_{3,1}$
0	0	0	0	0	4	4	3	3	2	2	1	1	0	0	0	0	0	0	$= 20$	$F_{2,2}$
0	0	0	0	0	1	0	2	1	3	2	4	3	4	0	0	0	0	0	$= 4$	$F_{3,2}$
0	0	0	0	0	0	0	0	0	0	0	0	0	0	2	1	1	0	0	$= 20$	$F_{2,3}$
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	2	1	$= 12$	$F_{3,3}$

Table 11: Case 6A, Partial Third Refinement

Case 6B																				
1	0	0	0	0	6	3	3	1	1	0	0	0	0	1	0	0	0	0	≤ 36	J_2
0	0	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	≤ 1	$J_{1,4}$
2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	$= 9$	$F_{2,1}$
0	1	0	2	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	$= 5$	$F_{3,1}$
0	0	0	0	0	4	3	3	2	2	1	1	0	0	0	0	0	0	0	$= 18$	$F_{2,2}$
0	0	0	0	0	1	2	1	3	2	4	3	5	4	0	0	0	0	0	$= 5$	$F_{3,2}$
0	0	0	0	0	0	0	0	0	0	0	0	0	0	2	1	1	0	0	$= 18$	$F_{2,3}$
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	2	1	$= 15$	$F_{3,3}$

Table 12: Case 6B, Partial Third Refinement

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