More on Regular Linear Spaces

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ABSTRACT

We extend the enumeration of regular linear spaces in [1] to at most 19 points. In addition, one of the 5 missing cases in the previous list is settled. The number of regular linear spaces of type $(15|2^{15}, 3^{30})$ is $10, 177, 328$.

1. INTRODUCTION

A *linear space* is a point line incidence geometry with the property that every pair of points is on exactly one line and every line has at least two points. We consider finite linear spaces, i.e. linear spaces with a finite number of points. In [2], the current authors enumerated all linear spaces of order at most 12. More specific types of linear spaces have also been studied. We denote the set of points (resp. lines) by $\mathcal{P}$ (resp. $\mathcal{B}$). We denote the incidence structure as a pair $(\mathcal{P}, \mathcal{B})$, and we identify a line with the set of points on it. We also call a line *block*, using the language of Design Theory, which is a related area.

A linear space is *regular* (in the sense of [1]) if for every integer i, the number of lines of size i through a point is independent of the point. Let a_i be the number of lines of size i through a point, for $2 \leq i \leq v$, where v is the total number of points. The number of lines of length i (also called i-lines) is $b_i = va_i/i$. We denote the type of a regular linear space as $(v|2^{b_2}, \ldots, v^{b_v})$, possibly omitting terms with $b_i = 0$. Previously, in [1], the parameter type $(v|2^{b_2}, 3^{b_3}, \ldots, v^{b_v})$ has been denoted as $(v|b_2, b_3, \ldots, b_v)$. That notation turned out to be impractical for larger parameter sets, hence the change.

In [1], the regular linear spaces on at most 16 points have been enumerated, with the exception of 5 parameter cases. The present note extends that enumeration to at most 19 points. In addition, the parameter case $(15|2^{15}, 3^{30})$ is settled.

For sake of completeness, we also mention that a linear space is called *proper* provided every line has at least three points. Proper linear spaces have been enumerated for up to 18 points in [5].

If possible, we will describe the relationship of regular linear spaces of a given type to other structures. Let us introduce some of the most important objects and constructions. For more details, we refer to the Handbook of Combinatorial Designs [9].

1. We let $\mathbb{P}_n(q)$ and $\mathbb{A}_n(q)$ be the projective and the affine spaces of dimension n over the field with q elements. If $n = 2$, they are 2-$(q^2 + q + 1, q + 1, 1)$ and 2-$(q^2, q, 1)$ designs, respectively. Any design with parameters of these two types if called a projective or affine plane (of order q), respectively. Note that for this we do not require q to be a prime power. However, no examples where q is not a prime power are known.

2. Every incidence structure $(\mathcal{P}, \mathcal{B})$ can be dualized as follows. Define a new incidence structure, the *dual*, as the pair $(\mathcal{P}', \mathcal{B}')$, where $\mathcal{P}' = \mathcal{B}$ and $\mathcal{B}' = \mathcal{P}$. For $B \in \mathcal{P}' = \mathcal{B}$ and $p \in \mathcal{B}' = \mathcal{P}$, we put

$$B \in p \iff p \in B,$$

where the first incidence is in $(\mathcal{P}', \mathcal{B}')$ and the second is in $(\mathcal{P}, \mathcal{B})$. So, roughly speaking we exchange points and lines and keep incidence the same.

3. A *configuration* of type $v_r b_k$ is a point line incidence structure with v points and b lines of size k such that every point is on r lines and any two points lie on at most one line. Configurations $v_r b_k$ with $k > 2$ are in one-to-one correspondence with regular linear spaces of type $(v|2^{b_2}, k^b)$ with $b_2 = a_2 = v - 1 - (k-1)b$. The projective and affine planes of order n are $(n^2+n+1)_{n+1}$ and $n^2_{n+1}(n^2+n)_n$ configurations, respectively. The dual of a projective plane is again a projective plane (of the same parameters). This is not true for affine planes.

4. A *Latin square* of order $n \geq 4$, denoted LSQ(n) gives rise to a regular linear space of type $(3n|3^{n^2}, n^3)$. The points consist of three sets of size n, which stand for rows, columns and digits of the Latin square. Let these be denoted $r_1, \ldots, r_n, c_1, \ldots, c_n$ and $d_1, \ldots, d_n$. The 3 n-blocks are $\{r_1, \ldots, r_n\}$, $\{c_1, \ldots, c_n\}$, and $\{d_1, \ldots, d_n\}$, The n^2 3-lines consist of one point of each type. More precisely, for each entry of the Latin square we declare a block as follows. If the digit in row i and column j of the Latin square is k, then the block is $\{r_i, c_j, d_k\}$. Conversely, any regular linear space with parameters $(3n|3^{n^2}, n^3)$ with $n \geq 4$ gives rise to a Latin square of order n. For example, the linear spaces $(12|3^{16}, 4^3)$ are the two Latin squares of order 4. We remark that under this correspondence, the isomorphism types of such linear spaces correspond to the "strong" isomorphism types of Latin squares, where rows, columns and digits may be exchanged. There are other concepts of isomorphism for Latin squares, which typically amount to limiting the possibilities of exchanging the three sets against each other (let "weak" isomorphism types be the Latin squares where all three sets must be fixed).

5. Several parameter cases will be identified as dual mesh as follows. Two disjoint lines in a linear space are called *parallel.* A *parallel class* is a set of pairwise parallel lines which together partition the set of points. A linear space on v points with lines of at most two sizes, k and possibly 2, is a *mesh,* denoted mesh(v,k,m), provided the k-lines form m parallel classes. Hence a mesh(v,k,m) is a regular linear space of type $(v|2^{b_2},k^{vm/k})$, for some b_2, but not every regular linear space of that type is a mesh. The parameter b_2 is determined as $b_2=\binom{v}{2}-\frac{vm}{k}\binom{k}{2}$. A mesh with $v=k^2$ is known as a *net.* A dual mesh is obtained as follows. For each of the parallel classes, adjoin a new point "at infinity" where all lines of that class intersect. Then remove the 2-lines and dualize the resulting incidence geometry (in the above sense). Finally, add sufficiently many 2-lines to turn the dual structure into a linear space. For example, a mesh$(30,5,3)$ is a regular linear space $(30|2^{255},5^{18})$ with 3 parallel classes. Dualizing in the above manner yields a $(18|2^{18},3^{30},6^{3})$. In general, the parameters of the dual mesh are

$$\left(\frac{vm}{k}\,\middle|\,2^{b_2'},m^v,\left(\frac{v}{k}\right)^m\right)$$

where

$$b_2'=\binom{\frac{vm}{k}}{2}-v\binom{m}{2}-m\binom{\frac{v}{k}}{2}.$$

2. TACTICAL DECOMPOSITIONS

Recall the concept of a *tactical configuration* of incidence structures as described for instance in [3]. Assume we have a finite incidence structure $(\mathcal{P},\mathcal{B})$, i.e. with both $\mathcal{P}$ and $\mathcal{B}$ finite. Assume further that $\mathcal{C}$ and $\mathcal{D}$ are partitions of $\mathcal{P}$ and $\mathcal{B}$, respectively. Let $\mathcal{C}$ have classes $C_1,\ldots,C_m$ and let $D_1,\ldots,D_n$ be the set of classes of $\mathcal{B}$. For $1\leq i\leq m$ and $1\leq j\leq n$, let

$$\alpha_{i,j}=\#\,B\in D_j:p\in B \tag{1}$$

for p in C_i, provided that this number exists, i.e. that the number of blocks of D_j containing a point of C_i is constant. Also, for $1\leq i\leq m$ and $1\leq j\leq n$, define

$$\beta_{i,j}=\#\,p\in C_i:p\in B \tag{2}$$

for $B\in B_j$, provided that this number exists. In case that all $\alpha_{i,j}$ exist, we call the incidence structure *row tactical.* In case that all $\beta_{i,j}$ exits, we call the incidence structure *column tactical.* An incidence structure which is both, row and column tactical is simply called *tactical.* For a tactical decomposition we always have

$$c_i\alpha_{i,j}=\beta_{i,j}d_j. \tag{3}$$

We may describe row tactical incidence structures by its scheme, which is the array

$$\begin{bmatrix} \rightarrow & d_1 & \cdots & d_n \\ c_1 & \alpha_{1,1} & & \alpha_{1,n} \\ \vdots & & & \\ c_m & \alpha_{m,1} & & \alpha_{m,n} \end{bmatrix} \tag{4}$$

(with $c_i = |C_i|$ and $d_j = |D_j|$ for $1 \le i \le m$ and $1 \le j \le n$.) We indicate the fact that this scheme is row tactical by a horizontal arrow in the top left corner. Also, a column tactical decomposition is described by the scheme

$$\begin{bmatrix} \downarrow & d_1 & \cdots & d_n \\ c_1 & \beta_{1,1} & & \beta_{1,n} \\ \vdots & & & \\ c_m & \beta_{m,1} & & \beta_{m,n} \end{bmatrix} \tag{5}$$

with a downward arrow indicating that the scheme is column tactical. We remark that dualizing (in the above sense) leads to the following exchange in parameters: the new $\alpha_{j,i}$ will equal the old $\beta_{i,j}$ and vice versa. Also, we note that there is no special notation for tactical decompositions (which are both row and column tactical). The reason for this is that the arrow indicates whether the decomposition scheme shows the $\alpha_{i,j}$ or the $\beta_{i,j}$. So, for a tactical decomposition we still need to make a choice as to whether we present it as a row tactical decomposition or as a column tactical decomposition.

Configurations have rather small tactical decompositions. A configuration $v_r b_k$ has $m = n = 1$ and furthermore $\alpha_{1,1} = r$, $\beta_{1,1} = k$, $c_1 = v$ and $d_1 = b$. The equation (3) reduces to $vr = kb$. The decompositions (4) and (5) are, respectively,

$$\begin{bmatrix} \rightarrow & b \\ v & r \end{bmatrix} \text{ and } \begin{bmatrix} \downarrow & b \\ v & k \end{bmatrix}$$

The dual structure has the decomposition

$$\begin{bmatrix} \rightarrow & v \\ b & k \end{bmatrix}$$

A configuration v_r is a configuration $v_r v_r$, i.e. with $k = r$ and $b = v$.

A mesh(v, k, m) has the tactical decomposition

$$\begin{bmatrix} \rightarrow & \frac{v}{k} & \cdots & \frac{v}{k} & b_2 \\ v & 1 & \cdots & 1 & 2b_2/v \end{bmatrix} \text{ or } \begin{bmatrix} \downarrow & \frac{v}{k} & \cdots & \frac{v}{k} & b_2 \\ v & k & \cdots & k & 2 \end{bmatrix}$$

The dual mesh has the decomposition

$$\begin{bmatrix} \downarrow & m & v & b_2' \\ \frac{vm}{k} & \frac{v}{k} & m & 2 \end{bmatrix}$$

A Latin square LSQ(n) has the tactical decomposition

$$\begin{bmatrix} \rightarrow & 1 & 1 & 1 & n^2 \\ n & 1 & 0 & 0 & n \\ n & 0 & 1 & 0 & n \\ n & 0 & 0 & 1 & n \end{bmatrix}$$

where the three groups of points of size n are the three sets corresponding to rows, column and digits as described above. This decomposition scheme would generate the weak isomorphism types of Latin squares, since the three groups of size n are fixed. On the other hand, the scheme

$$\begin{bmatrix} \rightarrow & 3 & n^2 \\ 3n & 1 & n \end{bmatrix}$$

would generate the strong isomorphism types. We will see later that finer decomposition schemes have advantages when trying to generate the incidence geometries from the scheme. In Section 4. we will discuss a method of refining the parameters of a decomposition scheme.

3. THE CONFIGURATIONS $\mathbf{15_6 30_3}$ A. K. A. REGULAR LINEAR SPACES OF TYPE $\mathbf{(15|2^{15}, 3^{30})}$

Regular linear spaces $(15|2^{15}, 3^{30})$ admit the tactical decomposition

$$\begin{bmatrix} \rightarrow & 15 & 30 \\ 15 & 2 & 6 \end{bmatrix} \tag{6}$$

or dually,

$$\begin{bmatrix} \rightarrow & 15 \\ 15 & 2 \\ 30 & 3 \end{bmatrix} \tag{7}$$

The lines of size 3 form a configuration$15_6 30_3$. Conversely, every configuration $15_6 30_3$ gives rise to a regular linear space $(15|2^{15}, 3^{30})$ by joining the 15 unconnected pairs of points with lines of size 2. This shows that regular linear spaces $(15|2^{15}, 3^{30})$ and configurations $15_6 30_3$ are equivalent objects. In this section (and in Section 5.) we will prove the following result:

Theorem 3.1. *The number of isomorphism types of regular linear spaces of type* $(15|2^{15}, 3^{30})$ *is* $10, 177, 328$.

Let us consider the dual configuration $15_6 30_3$ again. In the dual, i.e. in (7) we have 15 points of degree 2. They form a 2-regular graph on 15 points, or a configuration 15_2. Such a graph may be described by a partition of 15 into parts x_i of size at least 3, i.e. by an integer solution of the equation

$$15 = x_1 + x_2 + \cdots + x_\ell, \quad x_i \geq 3 \tag{8}$$

for some ℓ. Each 2-graph has a decomposition into cycles of length x_i satisfying (8). Conversely, each such partition describes a graph up to isomorphism. We let a_i be the number of parts of size i, so that (8) becomes

$$15 = \sum_{i=3}^{15} i a_i \tag{9}$$

There are 17 possible cases, which we call *cycle decompositions,* they are listed in Tab. I. We note that the automorphism group of a cycle decomposition is isomorphic to $\prod_{i=3}^{15} D_i \wr S_{a_i}$. The automorphism groups and their respective orders are indicated also.

TABLE I. The possible cycle decompositions of a 15_2

no.	partition	|Aut|	Aut
1	3^5	933120	$D_3 \wr S_5$
2	$3^3, 6$	15552	$(D_3 \wr S_3) \times D_6$
3	$3^2, 4, 5$	5760	$(D_3 \wr S_2) \times D_4 \times D_5$
4	$3^2, 9$	1296	$(D_3 \wr S_2) \times D_9$
5	$3, 4^3$	18432	$D_3 \times (D_4 \wr S_3)$
6	$3, 4, 8$	768	$D_3 \times D_4 \times D_8$
7	$3, 5, 7$	840	$D_3 \times D_5 \times D_7$
8	$3, 6^2$	1728	$D_3 \times (D_6 \wr S_2)$
9	$3, 12$	144	$D_3 \times D_{12}$
10	$4^2, 7$	1792	$(D_4 \wr S_2) \times D_7$
11	$4, 5, 6$	960	$D_4 \times D_5 \times D_6$
12	$4, 11$	176	$D_4 \times D_{11}$
13	5^3	6000	$D_5 \wr S_3$
14	$5, 10$	200	$D_5 \times D_{10}$
15	$6, 9$	216	$D_6 \times D_9$
16	$7, 8$	224	$D_7 \times D_8$
17	15	30	D_{15}

Let us investigate advantages and disadvantages of the previously mentioned decompositions in terms of their use for generating the geometries. We are referring to the generator program which has been written for [2]. This generator takes as input a decomposition scheme and produces all valid incidence matrices for that scheme up to isomorphism. Let us assume that we generate the spaces in a row-by-row (viz. point-by-point) fashion. Using (7), we would thus first create the 15 2-points. Clearly, this is a desirable approach, as we can access the geometries according to the 17 possible cycle decompositions of the 15_2. However, as we are in the dual setting, the condition that two points are on exactly one block is no longer present, which clearly weakens the strength of the generating process. (For sake of completeness, we should mention that we now have the additional condition that two blocks intersect in exactly one point, but that our current generator does not allow using this condition). On the other hand, if we generate from (6), then we could not divide cases according to the regular graphs.

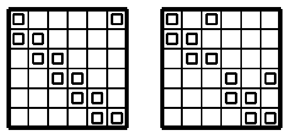
We remark that the isomorphism checking of the generator program usually respects the ordering of the classes of the decomposition. There are cases where this is not the best approach. For example, if the cycle decomposition has many parts of equal size then the group respecting the decomposition class-wise is $\prod_{i=3}^{15} D_i^{a_i}$. However, the true automorphism group of the 15_2 is $\prod_{i=3}^{15} D_i \wr S_{a_i}$, as seen. Hence there are cases in which more geometries than necessary are generated. On the other hand, there is a way of removing the decomposition altogether in the end and doing another isomorphism check. This last check would remove the isomorphic copies which are due to the fact that we did not start with the full configuration stabilizer in the first place.

3.1 Only short cycles (cases 1, 3, 5 and 13)

When generating cycles from a tactical decomposition scheme, we do not always get the expected results. For instance, the decomposition scheme

$$\begin{bmatrix} \rightarrow & 6 \\ 6 & 2 \end{bmatrix}$$

generates cycles of length 6 as well as double three cycles. This is because the scheme admits both cases, which can be seen from the two incidence matrices



Hence the presence of larger cycles may result in a mixing of cases, which is clearly undesirable. For cycles of length at most 5, this is not possible, so let us call these cycles *short*, and all other cycles *long*. Let us consider first the cases with only short cycles. In these cases, generating geometries based on refined decomposition schemes works well. As we want to generate pointwise, we start with (6) and refine it according to the cycle decomposition of the 2-blocks. The decomposition schemes in cases 1, 3, 5 and 13 are:

$$\#1: \begin{bmatrix} \rightarrow & 3 & 3 & 3 & 3 & 3 & 30 \\ 3 & 2 & 0 & 0 & 0 & 0 & 6 \\ 3 & 0 & 2 & 0 & 0 & 0 & 6 \\ 3 & 0 & 0 & 2 & 0 & 0 & 6 \\ 3 & 0 & 0 & 0 & 2 & 0 & 6 \\ 3 & 0 & 0 & 0 & 0 & 2 & 6 \end{bmatrix}, \#3: \begin{bmatrix} \rightarrow & 3 & 3 & 4 & 5 & 30 \\ 3 & 2 & 0 & 0 & 0 & 6 \\ 3 & 0 & 2 & 0 & 0 & 6 \\ 4 & 0 & 0 & 2 & 0 & 6 \\ 5 & 0 & 0 & 0 & 2 & 6 \end{bmatrix},$$

$$\#5: \begin{bmatrix} \rightarrow & 3 & 4 & 4 & 4 & 30 \\ 3 & 2 & 0 & 0 & 0 & 6 \\ 4 & 0 & 2 & 0 & 0 & 6 \\ 4 & 0 & 0 & 2 & 0 & 6 \\ 4 & 0 & 0 & 0 & 2 & 6 \end{bmatrix}, \#13: \begin{bmatrix} \rightarrow & 5 & 5 & 5 & 30 \\ 5 & 2 & 0 & 0 & 6 \\ 5 & 0 & 2 & 0 & 6 \\ 5 & 0 & 0 & 2 & 6 \end{bmatrix}$$

The above mentioned problem of generating too many isomorphic copies applies to case 1 in particular, since essentially we forget about the group S_5 acting on the set of 3-cycles. However, we do have the possibility of removing the decomposition in the end, and to do the isomorphism testing in a post-processing stage. We get the following computer result:

Proposition 3.2. *The number of regular linear spaces* $(15|2^{15}, 3^{30})$ *in cases* 1, 3, 5 *and* 13 *are* 146, 22816, 7515 *and* 27676, *respectively.*

At this point, one more remark is in order. Roughly speaking, a finer decomposition scheme leads to a better performance when generating the geometries for that scheme. One possible refinement would be to isolate a 3-cycle, which gives rise to the following tactical decomposition

$$\begin{bmatrix} \rightarrow & 3 & 12 & 18 & 12 \\ 3 & 2 & 0 & 6 & 0 \\ 12 & 0 & 2 & 3 & 3 \end{bmatrix}$$

This decomposition would yield all cases which involve a 3-cycle, i.e. cases 1 to 9. But we can also achieve decompositions for one particular cycle type. We show a row tactical decomposition for cycle type 5. This decomposition is not column tactical. However, merging all the point classes of size 4 results in a block tactical decomposition:

$$\#5: \begin{bmatrix} \rightarrow & 3 & 4 & 4 & 4 & 18 & 12 \\ 3 & 2 & 0 & 0 & 0 & 6 & 0 \\ 4 & 0 & 2 & 0 & 0 & 3 & 3 \\ 4 & 0 & 0 & 2 & 0 & 3 & 3 \\ 4 & 0 & 0 & 0 & 2 & 3 & 3 \end{bmatrix} \quad \rightsquigarrow \quad \begin{bmatrix} \rightarrow & 3 & 12 & 18 & 12 \\ 3 & 2 & 0 & 6 & 0 \\ 12 & 0 & 2 & 3 & 3 \end{bmatrix}$$

Of course, generating the dual scheme would allow us control about the cycle type which arises after 15 points.

3.2 One long cycle of length at most 9 (cases 2, 4, 6, 7, 10 and 11)

A modification of the refinement method presented above allows to handle the cases with exactly one long cycle of length at most 9. Take for instance the cycle type no. 2 which is $(3^3, 6)$. We refine in such a way that the long cycle comes first on the diagonal of the decomposition scheme:

$$\#2: \begin{bmatrix} \rightarrow & 6 & 3 & 3 & 3 & 30 \\ 6 & 2 & 0 & 0 & 0 & 6 \\ 3 & 0 & 2 & 0 & 0 & 6 \\ 3 & 0 & 0 & 2 & 0 & 6 \\ 3 & 0 & 0 & 0 & 2 & 6 \end{bmatrix}$$

At line 6 in the generation process, we check whether a cycle of length 6 has been generated, and reject the other cases. In a similar manner, we also handle the cases 4, 6, 7, 10 and 11, using the following refined row tactical decompositions:

$$\#4: \begin{bmatrix} \rightarrow & 9 & 3 & 3 & 30 \\ 9 & 2 & 0 & 0 & 6 \\ 3 & 0 & 2 & 0 & 6 \\ 3 & 0 & 0 & 2 & 6 \end{bmatrix}, \#6: \begin{bmatrix} \rightarrow & 8 & 4 & 3 & 30 \\ 8 & 2 & 0 & 0 & 6 \\ 4 & 0 & 2 & 0 & 6 \\ 3 & 0 & 0 & 2 & 6 \end{bmatrix}, \#7: \begin{bmatrix} \rightarrow & 7 & 5 & 3 & 30 \\ 7 & 2 & 0 & 0 & 6 \\ 5 & 0 & 2 & 0 & 6 \\ 3 & 0 & 0 & 2 & 6 \end{bmatrix}$$

$$\#10: \begin{bmatrix} \rightarrow & 7 & 4 & 4 & 30 \\ 7 & 2 & 0 & 0 & 6 \\ 4 & 0 & 2 & 0 & 6 \\ 4 & 0 & 0 & 2 & 6 \end{bmatrix}, \#11: \begin{bmatrix} \rightarrow & 6 & 5 & 4 & 30 \\ 6 & 2 & 0 & 0 & 6 \\ 5 & 0 & 2 & 0 & 6 \\ 4 & 0 & 0 & 2 & 6 \end{bmatrix}$$

We obtain the following computer result:

Proposition 3.3. *The number of regular linear spaces* $(15|2^{15}, 3^{30})$ *in cases* 2, 4, 6, 7, 10 *and* 11 *are* 7891, 102368, 184182, 174612, 82328 *and* 160684, *respectively.*

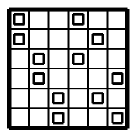
If the long cycle is very long (i.e. of length > 9), then we have to distinguish too many cases and therefore this method becomes unpractical.

3.3 Forcing a 6-cycle (cases 8 and 15)

Let us get back to the problem of generating a 6-cycle. If we take the input

$$\begin{bmatrix} \rightarrow & 3 & 3 \\ 6 & 1 & 1 \end{bmatrix}$$

then the 6-cycle splits into cycles of even length, but since a summand 2 is not possible, we get only one summand 6, i.e. the 6-cycle 1-2-5-6-4-3-1:



We can generate the cases 8 and 15 using the following decomposition. For case 8, we also use the 3-cycle. For case 15, we take the 9-cycle at the beginning and choose the proper cases at line 9.

$$\#8: \begin{bmatrix} \rightarrow & 3 & 3 & 3 & 3 & 3 & 18 & 12 \\ 3 & 2 & 0 & 0 & 0 & 0 & 6 & 0 \\ 6 & 0 & 1 & 1 & 0 & 0 & 3 & 3 \\ 6 & 0 & 0 & 0 & 1 & 1 & 3 & 3 \end{bmatrix}, \quad \#15: \begin{bmatrix} \rightarrow & 9 & 3 & 3 & 30 \\ 9 & 2 & 0 & 0 & 6 \\ 6 & 0 & 1 & 1 & 6 \end{bmatrix}$$

Of course, in case 8 we get geometries repeatedly since we have ordered the two 6-cycles. A post processing phase is necessary to eliminate isomorphic copies. We get the following computer result:

Proposition 3.4. *The number of regular linear spaces* $(15|2^{15}, 3^{30})$ *in cases* 8 *and* 15 *are* 83065 *and* 721962, *respectively.*

At this point, we are left with the remaining cases 9, 12, 14, 16 and 17. We will continue with these cases in Section 5.

4. REFINEMENT OF PARAMETERS

In [3], a method of calculating refined parameters algebraically has been presented. The idea is to create all possible refinements of a decomposition scheme, i.e. decompositions at a higher parameter depth. The hope is that the generation process runs more smoothly on the refined decompositions. Let us present this process for the parameters of regular linear spaces (for linear spaces in general, see [2]). Such a space admits a row tactical decomposition

$$\begin{bmatrix} \rightarrow & b \\ v & r \end{bmatrix} \tag{10}$$

where b is the total number of lines and r is the constant number of lines through a point. We call this the *parameters at depth 0.* The decomposition can be refined by computing the possible types of lines. Let b_i be the number of lines of length i in the space. Then

$$\binom{v}{2} = \sum_{i=2}^{v} b_i \binom{i}{2} \tag{11}$$

subject to the conditions that

$$v \mid ib_i \text{ for } i = 3, 4, \ldots, v. \tag{12}$$

We then have $b = \sum_{i=2}^{v} b_i$ and $r = \sum_{i=2}^{v} \frac{ib_i}{v}$. The *parameters at depth 1* are a column tactical decomposition of the form

$$\begin{bmatrix} \downarrow & b_2 & b_3 & \cdots & b_v \\ v & 2 & 3 & \cdots & v \end{bmatrix} \text{ or, dually } \begin{bmatrix} \rightarrow & v \\ b_2 & 2 \\ b_3 & 3 \\ \vdots & \vdots \\ b_v & v \end{bmatrix} \tag{13}$$

where $(b_2, \ldots, b_v)$ is a solution of (11). This decomposition may or may not be point tactical. If it is not point tactical, the procedure of [3] goes on and refines the decomposition of points to a tactical one. The whole process keeps refining the row and column decompositions alternately. Finiteness implies that this process will end eventually. In that case, we have obtained a tactical decomposition (which may be discrete). The even parameter depths correspond to point tactical decompositions, whereas the odd numbered parameter depths are block tactical decompositions.

Let us demonstrate the process of refining the parameters by an example. Consider one particular cycle decomposition of a $(15|2^{15}, 3^{30})$. We may start with the cycle type no. 14= $(5, 10)$, which gives us the following row tactical decomposition.

$$\#14: \begin{bmatrix} \rightarrow & 5 & 10 & 30 \\ 5 & 2 & 0 & 6 \\ 10 & 0 & 2 & 6 \end{bmatrix} \tag{14}$$

We call these the parameters at depth 2. Let $\mathcal{P}_i$ and $\mathcal{B}_j$ be the point and block classes, respectively, where $1 \leq i \leq 2$ and $1 \leq j \leq 3$. Hence we have $\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2$ and $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{B}_3$. The next step is to determine the refined line types. That is, for a line $B \in \mathcal{B}_j$ we compute all possible $(\beta_{1,j}, \beta_{2,j})$ with $\beta_{i,j}$ as in (2). The parameters at depth 3 describe how many lines of each refined line type exist. See [2] for more on how to compute refined line types. Briefly, in this case one proceeds as follows:

For a line B in $\mathcal{B}_1$ or $\mathcal{B}_2$ we must have a type $(2, 0)$ or $(0, 2)$, respectively. The possibilities for lines in $\mathcal{B}_3$ are

$$\mathbf{v}_1 = (0, 3),\ \mathbf{v}_2 = (1, 2),\ \mathbf{v}_3 = (2, 1)$$

(notice that the type $(3, 0)$ is impossible since each of the 3 points is on 2 2-lines from $\mathcal{B}_1$, which would require 6 distinct points in $\mathcal{P}_1$, in contradiction to $|\mathcal{P}_1| = 5$).

Let e_j be the number of lines of type $\mathbf{v}_j$. We are going to create linear equalities and inequalities as follows. The equation S_j counts the lines in $\mathcal{B}_j$. The equation

$F_{i,j}$ comes from double counting the number of flags in $\mathcal{P}_i \times \mathcal{B}_j$. The equation $J_{u,v}$ counts the joining of pairs of points from $\mathcal{P}_u$ and $\mathcal{P}_v$ (including the case $u = v$). Thus we have

$$\left.\begin{array}{rcrcrcrcl} & & e_1 & + & e_2 & + & e_3 & = & |\mathcal{B}_3| = 30 \quad (\mathrm{S}_3) \\ & & & & e_2 & + & 2e_3 & = & 5 \cdot 6 = 30 \quad (\mathrm{F}_{1,3}) \\ & & 3e_1 & + & 2e_2 & + & e_3 & = & 10 \cdot 6 = 60 \quad (\mathrm{F}_{2,3}) \\ 5 & & & & & + & e_3 & = & \binom{5}{2} = 10 \quad (\mathrm{J}_{1,1}) \\ & & & & 2e_2 & + & 2e_3 & = & 5 \cdot 10 = 50 \quad (\mathrm{J}_{1,2}) \\ 10 & + & 3e_1 & + & e_2 & & & = & \binom{10}{2} = 45 \quad (\mathrm{J}_{2,2}) \end{array}\right\} \tag{15}$$

($\mathrm{J}_{1,1}$) implies that $e_3 = 5$. From ($\mathrm{J}_{1,2}$) we infer that $e_2 = (50 - 10)/2 = 20$, so that $e_1 = 30 - 20 - 5 = 5$ by (S_3). This is in fact a solution, i.e. we have 5 lines of type $\mathbf{v}_1$, 20 lines of type $\mathbf{v}_2$ and 5 lines of type $\mathbf{v}_3$. Therefore, the unique block tactical decomposition at depth 3 is

$$\#14: \begin{bmatrix} \downarrow & 5 & 10 & 5 & 20 & 5 \\ 5 & 2 & 0 & 0 & 1 & 2 \\ 10 & 0 & 2 & 3 & 2 & 1 \end{bmatrix} \text{ or, dually } \begin{bmatrix} \rightarrow & 5 & 10 \\ 5 & 2 & 0 \\ 10 & 0 & 2 \\ 5 & 0 & 3 \\ 20 & 1 & 2 \\ 5 & 2 & 1 \end{bmatrix} \tag{16}$$

We may either construct the geometries from this scheme, or we may choose to calculate parameters at depth 4, which would lead to several possible decompositions. If constructing from a decomposition scheme at an odd parameter depth, we can extract the correct cycle decomposition at line 15 in the generation process.

For the purpose of illustrating the method in its full generality, let us go on and compute the parameters at depth 4. We denote by $\mathcal{P}_i$ and $\mathcal{B}_j$ the classes of point (resp. blocks) of the decomposition at depth 3. The next step is to determine the refined point types, i.e. the vectors $(\alpha_{i,1}, \ldots, \alpha_{i,5})$ for $i = 1, 2$, with $\alpha_{i,j}$ as in (1). The geometrical conditions of the linear space translate into a system of linear equation. There are two types of equations, called $\mathrm{C}_{i,j}$ and $\mathrm{S}_{i,j}$. Here, i and j are indices of the point classes $\mathcal{P}_i$ and block classes $\mathcal{B}_j$ of the parameters of depth 3. The equation $\mathrm{C}_{i,j}$ counts the connections of a point of $\mathcal{P}_i$ with points of $\mathcal{P}_j$ (if $i = j$ then we count connections of a point of $\mathcal{P}_i$ with *different* points of $\mathcal{P}_i$). The equation $\mathrm{S}_{i,j}$ forces the new type to be a refinement of the type in parameter depth two. For points of $\mathcal{P}_1$, we get the following system of equations, with $x_j = \alpha_{1,j}$ for $j = 1, \ldots, 5$.

$$\left.\begin{array}{rcrcrcrcrcl} x_1 & & & & & & & + & x_5 & = & 4 \quad (\mathrm{C}_{1,1}) \\ & & 2x_2 & + & 3x_3 & + & 2x_4 & + & x_5 & = & 10 \quad (\mathrm{C}_{1,2}) \\ & & & & x_3 & + & x_4 & + & x_5 & = & 6 \quad (\mathrm{S}_{1,3}) \\ \multicolumn{11}{c}{\text{subject to } x_1 = 2,\ x_2 = x_3 = 0} \end{array}\right\} \tag{17}$$

This system has the unique solution

$$\mathbf{v}_1 = (x_1, \ldots, x_5) = (2, 0, 0, 4, 2).$$

For points of $\mathcal{P}_2$, we have the following conditions (with $x_j = \alpha_{2,j}$ for $j = 1, \ldots, 5$)

$$\left.\begin{array}{rcl} 2x_1 + x_4 + 2x_5 &=& 5 \quad (\mathrm{C}_{2,1}) \\ x_2 + 2x_3 + x_4 &=& 9 \quad (\mathrm{C}_{2,2}) \\ x_3 + x_4 + x_5 &=& 6 \quad (\mathrm{S}_{2,3}) \\ \text{subject to } x_1 = 0,\ x_2 = 2 && \end{array}\right\} \tag{18}$$

There are exactly three solutions

$$\mathbf{v}_2 = (0,2,1,5,0),\ \mathbf{v}_3 = (0,2,2,3,1),\ \mathbf{v}_4 = (0,2,3,1,2).$$

For the parameters of depth 4, we need to know how many points of each type exist. This leads to the following system of linear equalities and inequalities. The inequalities come from considering intersections of lines of different types. Recall that in a linear space, two lines intersect in at most one point. Hence for disjoint sets of lines, the number of intersections between the two sets of lines is bounded above by the product of the respective numbers of lines in the sets. We let x_i be the number of points of type $\mathbf{v}_i$.

$$\left.\begin{array}{rclcll}
x_1 &=& 5 & & & (\mathrm{S}_1) \\
x_2 + x_3 + x_4 &=& 10 & & & (\mathrm{S}_2) \\
2x_2 + 2x_3 + 2x_4 &=& 20 & & & (\mathrm{F}_{2,2}) \\
x_2 + 2x_3 + 3x_4 &=& 15 & & & (\mathrm{F}_{2,3}) \\
5x_2 + 3x_3 + x_4 &=& 40 & & & (\mathrm{F}_{2,4}) \\
x_3 + 2x_4 &=& 5 & & & (\mathrm{F}_{2,5}) \\
x_1 &\le& \binom{5}{2} &=& 10 & (\mathrm{J}_{1,1}) \\
8x_1 &\le& 5 \cdot 20 &=& 100 & (\mathrm{J}_{1,4}) \\
4x_1 &\le& 5 \cdot 5 &=& 25 & (\mathrm{J}_{1,5}) \\
x_2 + x_3 + x_4 &\le& \binom{10}{2} &=& 45 & (\mathrm{J}_{2,2}) \\
2x_2 + 4x_3 + 6x_4 &\le& 10 \cdot 5 &=& 50 & (\mathrm{J}_{2,3}) \\
10x_2 + 6x_3 + 2x_4 &\le& 10 \cdot 20 &=& 200 & (\mathrm{J}_{2,4}) \\
2x_3 + 4x_4 &\le& 10 \cdot 5 &=& 50 & (\mathrm{J}_{2,5}) \\
x_3 + 3x_4 &\le& \binom{5}{2} &=& 10 & (\mathrm{J}_{3,3}) \\
5x_2 + 6x_3 + 3x_4 &\le& 5 \cdot 20 &=& 100 & (\mathrm{J}_{3,4}) \\
2x_3 + 6x_4 &\le& 5 \cdot 5 &=& 25 & (\mathrm{J}_{3,5}) \\
6x_1 + 10x_2 + 3x_3 &\le& \binom{20}{2} &=& 190 & (\mathrm{J}_{4,4}) \\
8x_1 + 3x_3 + 2x_4 &\le& 20 \cdot 5 &=& 100 & (\mathrm{J}_{4,5}) \\
x_1 + x_4 &\le& \binom{5}{2} &=& 10 & (\mathrm{J}_{5,5})
\end{array}\right\} \tag{19}$$

This system has 3 solutions

$$(x_1, x_2, x_3, x_4) \in \{(5,5,5,0), (5,6,3,1), (5,7,1,2)\}$$

corresponding to three point tactical decompositions at depth 4:

$$\begin{bmatrix} \rightarrow & 5 & 10 & 5 & 20 & 5 \\ 5 & 2 & 0 & 0 & 4 & 2 \\ 5 & 0 & 2 & 1 & 5 & 0 \\ 5 & 0 & 2 & 2 & 3 & 1 \end{bmatrix}, \begin{bmatrix} \rightarrow & 5 & 10 & 5 & 20 & 5 \\ 5 & 2 & 0 & 0 & 4 & 2 \\ 6 & 0 & 2 & 1 & 5 & 0 \\ 3 & 0 & 2 & 2 & 3 & 1 \\ 1 & 0 & 2 & 3 & 1 & 2 \end{bmatrix}, \begin{bmatrix} \rightarrow & 5 & 10 & 5 & 20 & 5 \\ 5 & 2 & 0 & 0 & 4 & 2 \\ 7 & 0 & 2 & 1 & 5 & 0 \\ 1 & 0 & 2 & 2 & 3 & 1 \\ 2 & 0 & 2 & 3 & 1 & 2 \end{bmatrix} \tag{20}$$

Of course, we can generate from these three decomposition schemes, or we may decide to compute the parameters of depth 5. We cannot show all the details here, but we remark that there are 33 block tactical decomposition schemes at depth 5. Of those, 18 come from the first scheme, 14 come from the second scheme, and exactly one refines the third scheme. We show only one of each type.

$$\begin{bmatrix} \downarrow & 5 & 10 & 5 & 5 & 5 & 10 & 5 \\ 5 & 2 & 0 & 0 & 1 & 1 & 1 & 2 \\ 5 & 0 & 1 & 1 & 0 & 1 & 2 & 0 \\ 5 & 0 & 1 & 2 & 2 & 1 & 0 & 1 \end{bmatrix},$$

$$\begin{bmatrix} \downarrow & 5 & 6 & 2 & 2 & 1 & 1 & 3 & 2 & 5 & 12 & 1 & 2 & 3 \\ 5 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 2 & 2 \\ 6 & 0 & 1 & 2 & 1 & 1 & 2 & 1 & 0 & 1 & 2 & 1 & 0 & 0 \\ 3 & 0 & 1 & 0 & 0 & 2 & 1 & 1 & 2 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix},$$

$$\begin{bmatrix} \downarrow & 5 & 4 & 4 & 2 & 1 & 2 & 2 & 2 & 15 & 3 & 4 & 1 \\ 5 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 2 & 2 \\ 7 & 0 & 1 & 2 & 1 & 1 & 2 & 1 & 1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 2 & 0 & 1 & 0 & 0 & 2 & 1 & 1 & 1 & 0 & 0 & 1 & 0 \end{bmatrix} \tag{21}$$

We have already pointed out that generating geometries from a rather fine decomposition is generally much easier than generating from a coarse decomposition. This is of course the main reason why one may want to compute the parameters at higher depths.

5. THE REMAINING CASES

In a similar fashion as in the previous section, we can handle cycle types no. 12 $= (4, 11)$ and no. $16 = (7, 8)$. We get the following computer result:

Proposition 5.1. *The number of regular linear spaces* $(15|2^{15}, 3^{30})$ *in cases* 12, 14 *and* 16 *are* 867368, 799714 *and* 698824, *respectively.*

The remaining cases are case no. $9 = (3, 12)$ and case no. $17 = (15)$. We generate blockwise using for case no. 9 also the special 3-cycle and choosing at line 15 the correct cycle type.

TABLE II. The Regular Linear Spaces $(15|2^{15}, 3^{30})$

no.	cycles	#	|Aut| statistic
1	3^5	146	$(1^{96}, 2^{19}, 3^8, 4^9, 5^1, 6^2, 8^3, 12^3, 18^1, 60^1, 72^2, 360^1)$
2	$3^3, 6$	7,891	$(1^{7677}, 2^{188}, 3^{17}, 4^2, 6^7)$
3	$3^2, 4, 5$	22,816	$(1^{22700}, 2^{116})$
4	$3, 3, 9$	102,368	$(1^{101980}, 2^{356}, 3^{20}, 6^{12})$
5	$3, 4^3$	7,515	$(1^{7117}, 2^{273}, 3^{33}, 4^{67}, 6^{11}, 8^4, 12^6, 24^4)$
6	$3, 4, 8$	184,182	$(1^{182978}, 2^{1149}, 4^{49}, 8^6)$
7	$3, 5, 7$	174,612	$(1^{174052}, 2^{560})$
8	$3, 6^2$	83,065	$(1^{82216}, 2^{759}, 3^{13}, 4^{56}, 6^6, 8^{11}, 12^3, 24^1)$
9	$3, 12$	1,004,029	$(1^{1001188}, 2^{2711}, 3^{70}, 4^{30}, 6^{24}, 8^4, 12^2)$
10	$4^2, 7$	82,328	$(1^{82092}, 2^{236})$
11	$4, 5, 6$	160,684	$(1^{160416}, 2^{268})$
12	$4, 11$	867,368	$(1^{866352}, 2^{1016})$
13	5^3	27,676	$(1^{27521}, 2^{61}, 3^{78}, 6^{14}, 10^1, 15^1)$
14	$5, 10$	799,714	$(1^{799030}, 2^{679}, 5^4, 10^1)$
15	$6, 9$	721,962	$(1^{721600}, 2^{318}, 3^{42}, 6^2)$
16	$7, 8$	698,824	$(1^{698248}, 2^{576})$
17	15	5,232,148	$(1^{5231982}, 3^{160}, 5^4, 15^2)$
		10,177,328	

Table II summarizes the number of spaces for each of the 17 cycle types, together with a statistic of the corresponding automorphism group orders. Thus we have proved that there are 10,177,328 regular linear spaces of type $(15|2^{15}, 3^{30})$.

6. SOME SPECIAL GEOMETRIES OF TYPE $(15|2^{15}, 3^{30})$

If we deflate the 5 3-cycles of no. 1 to 5 3-blocks, then we get Steiner systems on 15 points with a distinguished parallel class. Therefore the number 146 is the number of the parallel classes of all Steiner systems on 15 points, up to embedding isomorphism, [14], [13]. In particular, the configuration with automorphism group order 360 comes from the projective space $\mathbb{P}_3(2)$. Its collineation group $\mathrm{PGL}(4,2)$ of order 20160 acts transitively on the set of 56 parallel classes. So, the stabilizer has order $20160/56 = 360$.

Among the enormous number of geometries with cycle type (15), there are two species with a group of order 15. One can figure out directly what these configurations are. Anticipating a cyclic group of order 15 acting transitively, we need only use triples which meet their shifts in at most one point. If we denote the points by $0, 1, 2, 3, ..., 14$ and describe the triple by the three differences $\neq 1, 2$, then only the following triples are possible:

$$(2,6,7), (2,8,5), (2,4,9), (2,3,10), (3,5,7), (3,4,8), (4,5,6).$$

We need two of them in order to get $15 + 15 = 30$ triples, so we must choose a compatible pair of such triples. There is only one combination possible, namely $2, 4, 9$ with $3, 5, 7$. Since we may allow a reflection for one of these triples we get the following two configurations with cyclic group C_{15}:

$$\{(0,2,6),(0,3,8)\}^{C_{15}} \text{ and } \{(0,2,6),(0,3,10)\}^{C_{15}}.$$

Another geometry with automorphism group order 15 is under those with cycle type (5^3). If one tries to see how the triples might be arranged with respect to the 3 cycles, one gets the idea that there should be the following tactical decomposition:

$$\begin{bmatrix} \rightarrow & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 \\ 5 & 2 & 0 & 0 & 2 & 1 & 0 & 1 & 1 & 1 \\ 5 & 0 & 2 & 0 & 0 & 2 & 1 & 1 & 1 & 1 \\ 5 & 0 & 0 & 2 & 1 & 0 & 2 & 1 & 1 & 1 \end{bmatrix}$$

This decomposition admits 6 geometries, and one of these has indeed an automorphism group of order 15 acting transitively on the point set. There are two orbits on the 30 triples, one consisting of triples which have two points in one 5-cycle, and the other consisting of triples having the three points distributed to all three cycles. If we denote the points by $\{0,1,2,...,14\}$ according to the cyclic group C_{15}, then we can describe this linear space as:

$$\{(0,3,4),(0,5,13)\}^{C_{15}}.$$

The three 5-cycles are in this notation:

$$(0,3,6,9,12),\ (1,4,7,10,13) \text{ and } (2,5,8,11,14).$$

If the automorphism group is transitive on the point set, its order must be divisible by 15. Besides the examples just given, there is only one other case with this condition: the space with cycle type 3^5 and group order 60. Looking at the collineation group, we see a C_5-action on the set of the 5 3-cycles. The stabilizer on one 3-cycle acts as a C_4-action on the other 4 cycles. So this is the group of the affine line over $GF(5)$ having order 20. In addition there is the kernel C_3 which maps each of the 3-cycles onto itself.

With help of the C_{15}-cycle $(0,1,2,...,14)$ one may describe the blocks as follows:

$$\{(0,11,14),(0,7,9)\}^{C_{15}}.$$

The 5 3-cycles are

$$(0,5,10),\ (1,6,11),\ (2,7,12),\ (3,8,13) \text{ and } (4,9,14).$$

7. REGULAR LINEAR SPACES ON AT MOST 19 POINTS

In Table III, we present the extended list of regular linear spaces on up to 19 points. A few comments are in order. Still, several parameter cases are unsettled. All we can say is that each parameter case in the list is realizable, i. e. there exists at least one linear space.

8. ACKNOWLEDGMENT

The authors would like to thank the two referees for helpful comments.

TABLE III. Regular Linear Spaces on 13 to 16 Points

parameter	#	comment
$(2\|1)$	1	one line
$(3\|2)$	1	one line
$(3\|2^3)$	1	$\binom{3}{2}$
$(4\|4)$	1	one line
$(4\|2^6)$	1	$\binom{4}{2}$
$(5\|5)$	1	one line
$(5\|2^{10})$	1	$\binom{5}{2}$
$(6\|6)$	1	one line
$(6\|2^3, 3^4)$	1	$\binom{4}{2}$ dually
$(6\|2^9, 3^2)$	1	2 disjoint 3-lines
$(6\|2^{15})$	1	$\binom{6}{2}$
$(7\|7)$	1	one line
$(7\|3^7)$	1	$\mathbb{P}_2(2)$ or configurations 7_3
$(7\|2^2 1)$	1	$\binom{7}{2}$
$(8\|8)$	1	one line
$(8\|2^4, 3^8)$	1	derived $\mathbb{A}_2(3)$ or configurations 8_3
$(8\|2^{16}, 4^2)$	1	2 disjoint 4-lines
$(8\|2^{28})$	1	$\binom{8}{2}$
$(9\|9)$	1	one line
$(9\|3^{12})$	1	$\mathbb{A}_2(3)$
$(9\|2^9, 3^9)$	3	configurations 9_3
$(9\|2^{18}, 3^6)$	2	cubic graphs of order 6, dually, or derived LSQ(4)
$(9\|2^{27}, 3^3)$	1	3 disjoint 3-lines
$(9\|2^{36})$	1	$\binom{9}{2}$
$(10\|10)$	1	one line
$(10\|2^{15}, 4^5)$	1	$\binom{5}{2}$ dually
$(10\|2^{15}, 3^{10})$	10	configurations 10_3 [8]
$(10\|2^{25}, 5^2)$	1	two disjoint 5-lines
$(10\|2^{45})$	1	$\binom{10}{2}$
$(11\|11)$	1	one line
$(11\|2^{22}, 3^{11})$	31	configurations 11_3
$(11\|2^{55})$	1	$\binom{11}{2}$
$(12\|12)$	1	one line
$(12\|3^4, 4^9)$	1	derived $\mathbb{P}_2(3)$
$(12\|3^{16}, 4^3)$	2	LSQ(4)
$(12\|2^6, 3^8, 4^6)$	1	
$(12\|2^6, 3^{20})$	5	derived STS(13)
$(12\|2^{12}, 4^9)$	1	dual of $\mathbb{A}_2(3)$
$(12\|2^{12}, 3^{12}, 4^3)$	4	configurations 12_3 a with parallel class, or derived LSQ(5)
$(12\|2^{18}, 3^4, 4^6)$	1	

TABLE III. (continued)

parameter	#	comment
$(12\|2^{18},3^{16})$	574	configurations $12_4 16_3$
$(12\|2^{24},3^{8},4^{3})$	8	cubic graphs of order 8 with a parallel class, dually
$(12\|2^{30},4^{6})$	1	4-regular graphs on 6 points, dually
$(12\|2^{30},3^{12})$	229	configurations 12_3
$(12\|2^{36},6^{2})$	1	2 disjoint 6-lines
$(12\|2^{36},3^{4},4^{3})$	1	3×4 grid
$(12\|2^{42},3^{8})$	6	cubic graphs of order 8, dually
$(12\|2^{48},4^{3})$	1	3 disjoint 4-lines
$(12\|2^{54},3^{4})$	1	4 disjoint 3-lines
$(12\|2^{66})$	1	$\binom{12}{2}$
$(13\|13)$	1	one line
$(13\|4^{13})$	1	$\mathbb{P}_2(4)$ or configurations 13_4
$(13\|3^{26})$	2	STS(13)
$(13\|2^{39},3^{13})$	2036	configurations 13_3[11]
$(13\|2^{78})$	1	$\binom{13}{2}$
$(14\|14)$	1	one line
$(14\|2^{7},4^{14})$	1	configurations 14_4 [4]
$(14\|2^{7},3^{28})$	787	derived STS(15)
$(14\|2^{49},7^{2})$	1	2 disjoint 7-lines
$(14\|2^{49},4^{7})$	2	
$(14\|2^{49},3^{14})$	21399	configurations 14_3
$(14\|2^{91})$	1	$\binom{14}{2}$
$(15\|15)$	1	one line
$(15\|3^{5},4^{15})$	1	derived $\mathbb{A}_2(4)$
$(15\|3^{15},5^{6})$	1	$\binom{15}{2}$+15 parallel classes 2+2+2 (dually)
$(15\|3^{25},5^{3})$	2	LSQ(5)
$(15\|3^{35})$	80	STS(15)
$(15\|2^{15},4^{15})$	4	configurations 15_4 [4]
$(15\|2^{15},3^{10},5^{6})$	1	
$(15\|2^{15},3^{20},5^{3})$	40	configurations 15_3 with a parallel class, or derived LSQ(6)
$(15\|2^{15},3^{30})$	10,177,328	configurations $15_6 30_3$ Section 3.
$(15\|2^{30},3^{5},5^{6})$	1	
$(15\|2^{30},3^{15},5^{3})$	251	
$(15\|2^{30},3^{25})$		configurations $15_5 25_3$
$(15\|2^{45},5^{6})$	1	$\binom{6}{2}$ dually
$(15\|2^{45},3^{10},5^{3})$	23	
$(15\|2^{45},3^{20})$		configurations $15_4 20_3$
$(15\|2^{60},3^{5},5^{3})$	1	3×5 grid
$(15\|2^{60},3^{15})$	245342	configurations 15_3
$(15\|2^{75},5^{3})$	1	3 disjoint 5-lines
$(15\|2^{75},3^{10})$	21	cubic graphs on 10 points, dually
$(15\|2^{90},3^{5})$	1	5 disjoint 3-lines
$(15\|2^{105})$	1	$\binom{15}{2}$

TABLE III. (continued)

parameter	#	comment
$(16\|16)$	1	one line
$(16\|4^{20})$	1	$\mathbb{A}_2(4)$
$(16\|3^{16}, 4^{12})$	1	
$(16\|3^{32}, 4^{4})$	23	see [10]
$(16\|2^{24}, 4^{16})$	19	configurations 16_4 [4]
$(16\|2^{24}, 3^{16}, 4^{8})$	300880	
$(16\|2^{24}, 3^{32})$		configurations $16_6 32_3$
$(16\|2^{48}, 4^{12})$	574	configurations $16_3 12_4$
$(16\|2^{48}, 3^{16}, 4^{4})$	88	configurations 16_3 with a parallel class
$(16\|2^{64}, 8^{2}2)$	1	2 disjoint 8-lines
$(16\|2^{72}, 4^{8})$	6	4-regular graphs on 8 points, dually
$(16\|2^{72}, 3^{16})$	3,004,881	configurations 16_3 [6]
$(16\|2^{96}, 4^{4})$	1	four disjoint 4-lines
$(16\|2^{120})$	1	$\binom{16}{2}$
$(17\|17)$	1	one line
$(17\|2^{34}, 4^{17})$	1972	configurations 17_4 [4]
$(17\|2^{34}, 3^{34})$		
$(17\|2^{85}, 3^{17})$	38,904,499	configurations 17_3 [6]
$(17\|2^{136})$	1	$\binom{17}{2}$
$(18\|18)$	1	one line
$(18\|3^{36}, 6^{3})$	12	LSQ(6) = dual mesh(36, 6, 3) [7]
$(18\|2^{9}, 3^{12}, 4^{18})$	77	
$(18\|2^{9}, 3^{30}, 4^{9})$		
$(18\|2^{9}, 3^{48})$		
$(18\|2^{18}, 3^{30}, 6^{3})$	4.260	dual mesh(30, 5, 3)
$(18\|2^{27}, 3^{6}, 4^{18})$		
$(18\|2^{27}, 3^{24}, 4^{9})$		
$(18\|2^{27}, 3^{42})$		
$(18\|2^{36}, 3^{24}, 6^{3})$		dual mesh(24, 4, 3)
$(18\|2^{45}, 4^{18})$	971, 171	configurations 18_4 [4]
$(18\|2^{45}, 3^{18}, 4^{9})$		
$(18\|2^{45}, 3^{36})$		
$(18\|2^{54}, 3^{18}, 6^{3})$	568	dual mesh(18, 3, 3)
$(18\|2^{63}, 3^{12}, 4^{9})$		
$(18\|2^{63}, 3^{30})$		
$(18\|2^{72}, 3^{12}, 6^{3})$	157	dual mesh(12, 2, 3)
$(18\|2^{81}, 9^{2})$	1	
$(18\|2^{81}, 3^{6}, 4^{9})$	150, 373	
$(18\|2^{81}, 3^{24})$		
$(18\|2^{90}, 3^{6}, 6^{3})$	1	3×6 grid
$(18\|2^{99}, 4^{9})$	16	4-regular graphs on 9 points, dually
$(18\|2^{99}, 3^{18})$	530,425,205	configurations 18_3 [6]
$(18\|2^{108}, 6^{3})$	1	3 6-lines
$(18\|2^{117}, 3^{12})$	94	3-regular graphs on 12 points, dually

TABLE III. (continued)

parameter	#	comment
$(18\|2^{135}, 3^6)$	1	6 3-lines
$(18\|2^{153})$	1	$\binom{18}{2}$
$(19\|19)$	1	one line
$(19\|3^{19}, 4^{19})$	56	
$(19\|3^{57})$	$11,084,874,829$	STS(19) [12]
$(19\|2^{57}, 4^{19})$		configurations 19_4
$(19\|2^{57}, 3^{38})$		
$(19\|2^{114}, 3^{19})$		configurations 19_3
$(19\|2^{171})$	1	$\binom{19}{2}$

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Received
Accepted