On a theorem of Wielandt for finite primitive permutation groups *

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Dedicated to the memory of Helmut Wielandt

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Abstract

Let G be a finite primitive permutation group with a non-trivial, non-regular normal subgroup N, and let Γ be an orbit of a point stabiliser N_α. Then each composition factor S of N_α occurs as a section of the permutation group induced by N_α on Γ. The case $N = G$ is a theorem of Wielandt. The general result and some of its corollaries are useful for studying automorphism groups of combinatorial structures.

1 Introduction

In his book on finite permutation groups Helmut Wielandt proved [6, Theorem 18.2] that each composition factor of a point stabiliser in a finite primitive permutation group is involved in the action on each of the non-trivial orbits of the stabiliser. The purpose of this paper is to generalise this result and several of its corollaries. Our investigations were motivated by a geometric application to finite linear spaces (see [5, Proposition 5.2]).

First we review some of the basic theory about transitive permutation groups. The proofs of these assertions may be found in [6]. Let G be a transitive permutation group on a finite set Ω and let $\alpha \in \Omega$. If H is a subgroup containing the stabiliser G_α of α, then the H-orbit $B = \{\alpha^h \mid h \in H\}$ containing α is a block of imprimitivity for G, that is, for all $g \in G$, either $B = B^g$ or $B \cap B^g = \emptyset$. Moreover all blocks of imprimitivity containing α are of this form. The group G is *primitive* if the only blocks of imprimitivity containing α are the trivial blocks $\{\alpha\}$ and Ω, or equivalently, if G_α is maximal in G. For a block of imprimitivity B, the set of images of B under elements of G forms a partition $\mathcal{P}$ of Ω that is G-invariant in the sense that for each $C \in \mathcal{P}$ and $g \in G$ the image $C^g \in \mathcal{P}$. If N is a normal subgroup of G then the N-orbits are blocks of imprimitivity permuted transitively by G, with the N-orbit containing α being the orbit of the subgroup $H = G_\alpha N$. Thus if G is primitive, then a non-trivial normal subgroup is transitive.

A transitive permutation group G is called *regular* if $G_\alpha = 1$, and otherwise is said to be *non-regular*. If $U < G$ and U fixes setwise a subset $\Gamma \subseteq \Omega$ then U induces a permutation group on Γ, denoted U^Γ, and $U^\Gamma \cong U/U_{(\Gamma)}$, where $U_{(\Gamma)}$ denotes the pointwise stabiliser of Γ in U.

2 A generalisation of a result of Wielandt

Wielandt's theorem [6, Theorem 18.2] can be generalized as follows (his result is the case where $N = G$):

Theorem 2.1 *Let G be a primitive permutation group on a finite set Ω, with a non-trivial, non-regular normal subgroup N. Let $\alpha \in \Omega$, and let Γ be an orbit of N_α in $\Omega \setminus \{\alpha\}$. If S is a composition factor of N_α, then there exists $U \leq N_\alpha$ such that S is a composition factor of the permutation group U^Γ induced by U on Γ.*

Proof. Suppose that G is a primitive permutation group on a finite set Ω, with a non-trivial, non-regular normal subgroup N. Let $\alpha \in \Omega$, let Γ be an orbit of N_α in $\Omega \setminus \{\alpha\}$, and let Δ be the G_α-orbit containing Γ.

Let S be a composition factor of N_α, and let $V \leq N_\alpha$ be chosen minimally with respect to inclusion such that S is a composition factor of V. By [6, Proposition 18.1], there exists $g \in G$ such that $U_0 := g^{-1}Vg \leq G_\alpha$ and $U_0^\Delta \neq 1$. Since N is normal in G it follows that $U_0 \leq N \cap G_\alpha = N_\alpha$, and since $U_0^\Delta \neq 1$ we must have $U_0^{\Gamma'} \neq 1$ for some N_α-orbit Γ' contained in Δ. Now N_α is normal in G_α and therefore G_α acts transitively on the set of N_α-orbits in Δ. Thus there is some $x \in G_\alpha$ such that $(\Gamma')^x = \Gamma$. Then $U := U_0^x \leq N_\alpha$ and the permutation group induced by U on Γ is $U^\Gamma = (U_0^{\Gamma'})^x \neq 1$. Thus the pointwise stabiliser $U_{(\Gamma)}$ of Γ in U is a proper normal subgroup of U and $U^\Gamma \cong U/U_{(\Gamma)}$. We will prove that S is a composition factor of U^Γ. Since U is conjugate to V, S is a composition factor of U, but if S were also a composition factor of $U_{(\Gamma)}$, then S would be a composition factor of the proper normal subgroup $gxU_{(\Gamma)}x^{-1}g^{-1}$ of V, contradicting the minimality of V. Therefore S is not a composition factor of $U_{(\Gamma)}$, and hence S is a composition factor of $U^\Gamma \cong U/U_{(\Gamma)}$.

A series of corollaries follow immediately which are analogous to [6, Theorems 18.3, 18.4, 18.5]. By N_α^Γ we mean the permutation group on Γ induced by N_α.

Corollary 2.2 *Let $G, N, \Omega, \alpha, \Gamma$ be as in Theorem 2.1. Then the following hold.*

(a) The only fixed point of N_α in Ω is α.

(b) If N_α^Γ is soluble then N_α is soluble.

(c) If a prime p divides $|N_\alpha|$, then p also divides $|N_\alpha^\Gamma|$.

(d) If N_α^Γ is a p-group then also N_α is a p-group.

Wielandt proves an additional theorem [6, Theorem 18.7] on this theme, namely that if $N = G$ in Theorem 2.1, and $|\Gamma| = 2$, then G is a Frobenius group; and he mentions [6, Exercise 18.8] that application of a result of Miller (1899) implies further that G is a dihedral group of twice prime order. Although this is no longer true when $N \neq G$, we can determine all possibilities for this situation. However the proof of this more general result depends on the finite simple group classification. We denote by $\mathrm{Soc}(G)$ the *socle* of a group G, that is, the product of its minimal normal subgroups.

Theorem 2.3 *Let G be a primitive permutation group on a finite set Ω of size n, and suppose that a normal subgroup N of G is such that, for $\alpha \in \Omega$, N_α has an orbit of length 2 in Ω. Then one of the following holds.*

(a) $\mathrm{Soc}(G) = \mathrm{Soc}(N) = Z_p^\ell$ and $n = p^\ell$, for some odd prime p and positive integer ℓ. Moreover N_α is an elementary abelian 2-group.

(b) $\mathrm{Soc}(G) = \mathrm{Soc}(N) = \mathrm{L}_2(q)^\ell$ and $(n, q) = (21^\ell, 7)$ or $(45^\ell, 9)$ for some positive integer ℓ. Moreover $G \leq H \wr S_\ell$, in product action on $\Omega = \Delta^\ell$ and if $q = 7$ then $H = \mathrm{PGL}_2(7)$ while if $q = 9$ then $H = \mathrm{PGL}_2(9)$, M_{10} or $\mathrm{P\Gamma L}_2(9)$.

Proof. Suppose that G is a primitive permutation group on a finite set Ω, with a normal subgroup N such that, for $\alpha \in \Omega$, N_α has an orbit Γ of length 2. Let Δ be the G_α-orbit containing Γ. By Corollary 2.2, N_α is a 2-group, and fixes only the point α. Thus each N_α-orbit Γ' in $\Omega \setminus \{\alpha\}$ has length greater than 1 and the group induced by N_α on Γ' is therefore a non-trivial 2-group, whence $|\Gamma'|$ is even. It follows that $|\Omega|$ is odd and N_α is a Sylow 2-subgroup of N. It follows from the O'Nan Scott Theorem [2, Theorem 4.1A] that a finite primitive permutation group G having a non-regular normal subgroup N with N_α a Sylow 2-subgroup is either of affine type, or almost simple or of product action type.

In the first case $|\Omega| = p^\ell$ for some odd prime p and positive integer ℓ, and $G = KG_0$ where $K = \mathrm{Soc}(G) = \mathrm{Soc}(N) = Z_p^\ell$ and G_0 is an irreducible subgroup of $\mathrm{GL}_\ell(p)$. We may identify Ω with an ℓ-dimensional vector space over $\mathrm{GF}(p)$, and α with the zero vector so that $G_\alpha = G_0$. Now, since $G_\alpha = G_0$ is irreducible, the G_α-orbit Δ containing Γ spans Ω and so G_α acts faithfully on Δ. Since N_α is normal in G_α, each of the N_α-orbits in Δ has length 2, and it follows that N_α acts faithfully on Δ as an elementary abelian 2-group.

If G is primitive in product action then $G \leq H \wr S_\ell$ in product action on $\Omega = \Omega_0^\ell$ where $\ell \geq 2$, H is almost simple and primitive on Ω_0, and $\mathrm{Soc}(G) = \mathrm{Soc}(N) = \mathrm{Soc}(H)^\ell$. If G is almost simple then set $H = G$, $\Omega_0 = \Omega$, and $\ell = 1$. The possibilities for (H, Ω_0) primitive and of odd degree were classified independently in [3] and [4]. We deduce from these classifications that the only possibilities, such that a point stabiliser in a normal subgroup is a 2-group, are $\mathrm{Soc}(H) = \mathrm{L}_2(q)$ where $q = 2^a \pm 1$ for some a. In this latter case, $N_\alpha = D_{q\pm1}^\ell = D^\ell$, say, and we may take α as an ℓ-tuple $(\delta, \ldots, \delta) \in \Omega_0^\ell$. A point β in the N_α-orbit Γ of length 2 is of the form $\beta = (\delta_1, \ldots, \delta_\ell) \in \Omega_0^\ell$, and since $\mathrm{Soc}(H)^\ell$ is transitive on Ω_0^ℓ, $\delta_i = \delta^{x_i}$ for some $x_i \in \mathrm{L}_2(q)$, and $N_{\alpha\beta} = \prod_{i=1}^{\ell}(D \cap D^{x_i})$. Since $N_{\alpha\beta}$ is a proper subgroup of N_α, there exists i such that $D^{x_i} \neq D$, and for such an i we have that $D \cap D^{x_i} \leq Z_2 \times Z_2$, of index at least 2^{a-2} in D. It follows that $a \leq 3$, and so $q = 5, 7$ or 9. However if $q = 5$ then $H \cong A_5$ or S_5 and there is no maximal subgroup of H which intersects A_5 in a Sylow 2-subgroup. Similar considerations for $q = 7, 9$ yield that either $q = 7$, $H = \mathrm{PGL}_2(7)$ and $|\Omega| = 21^\ell$, or $q = 9$, $H = \mathrm{PGL}_2(9)$, M_{10} or $\mathrm{P\Gamma L}_2(9)$ and $|\Omega| = 45^\ell$.

It is easy to construct examples of groups G satisfying the hypotheses of Theorem 2.3 with $\mathrm{Soc}(N)$ elementary abelian. In the next section we provide some information about the groups in Theorem 2.3 (b).

3 The groups in Theorem 2.3 (b)

We finish the paper with a brief discussion of the structure of the groups satisfying part (b) of Theorem 2.3, including information about some of the N_α-orbits.

Suppose that G is a primitive permutation group on a finite set Ω of size n, satisfying Theorem 2.3 (b). We showed in the proof that

$$\mathrm{Soc}(N) = L^\ell \leq G \leq H \wr S_\ell$$

in product action on $\Omega = \Omega_0^\ell$ of degree $n = n_0^\ell$, where $L = \mathrm{L}_2(q)$, $\ell \geq 1$, $(n_0, q) = (21, 7)$ or $(45, 9)$, and $H = \mathrm{PGL}_2(7)$ if $q = 7$, or $\mathrm{L}_2(9) < H \leq \mathrm{P\Gamma L}_2(9)$, $H \not\leq S_6$, if $q = 9$. Take $\alpha = (\delta, \ldots, \delta) \in \Omega$, where $\delta \in \Omega_0$, let Δ be an N_α-orbit of length 2 in Ω, and let $\mathcal{D} = \{\Delta^g \mid g \in G\}$. First we give some information about the L-action on Ω_0, that demonstrates in particular that N_α has orbits of length 2 in the case $\ell = 1$. The assertions may be easily checked by hand. Alternatively they may be checked using a group theoretic computer package. (The check was conducted by the authors using DISCRETA [1].)

Lemma 3.1 **(a)** *The group L_δ has orbit lengths in Ω_0 as follows:* $1, 2, 2, 4, 4, 8$ *(if $q = 7$) or* $1, 2, 2, 4, 4, 8, 8, 8, 8$ *(if $q = 9$).*

(b) *The union of δ and an L_δ-orbit of length 2 is a block of imprimitivity for L in Ω_0. The group H_δ interchanges the two L_δ-orbits of length 2, and the two L_δ-orbits of length 4, and if $q = 9$ then H_δ also interchanges at least two of the L_δ-orbits of length 8.*

(c) *All of the L_δ-orbits of length 2 or of length 4, and also, in the case $q = 9$, two of the L_δ-orbits of length 8 are unions of pairwise disjoint elements of $\mathcal{D}$.*

Now we use this information to explore the general case $\ell \geq 1$.

Lemma 3.2 **(a)** *The group N leaves invariant exactly 2ℓ partitions of Ω with parts of size 3 and these are permuted transitively by G.*

(b) *The group N_α has exactly 2ℓ orbits of length 2 in C and these are permuted transitively by G_α.*

(c) *Moreover $|\mathcal{D}| = 2\ell n$, and $\{\alpha, \beta, \gamma\}$ is a block of imprimitivity for N if and only if $\{\beta, \gamma\} \in \mathcal{D}$ and is an N_α-orbit.*

Proof. By Lemma 3.1, $L = \mathrm{L}_2(q)$ leaves invariant exactly two partitions of Ω_0 with parts of size 3, and these partitions are interchanged by H. Also a pair p of points from Ω_0 is an L_δ-orbit of length 2 if and only if $p \cup \{\delta\}$ is the part of one of these invariant partitions containing δ.

Let Σ be the block containing α of an N-invariant partition with $|\Sigma| = 3$, and let $\beta \in \Sigma \setminus \{\alpha\}$. Since $\beta \neq \alpha$, for some $i \leq \ell$ the i^{th}-entry β_i of β differs from δ. Since Σ contains α, the group N_α fixes Σ setwise, and hence Σ contains all points of the N_α-orbit containing β. Since $|\Sigma| = 3$ it follows that β_i lies in an L_δ-orbit of length 2 and $\beta_j = \delta$ for all $j \neq i$. Moreover, for each i and for each L_δ-orbit p of length 2 in Ω, the three points β such that $\beta_i \in \{\delta\} \cup p$, and $\beta_j = \delta$ for $j \neq i$, form a block of imprimitivity for N in Ω. Thus N leaves invariant exactly 2ℓ partitions of Ω with parts of size 3. Since G is a primitive subgroup of $\mathrm{Sym}(\Omega_0) \wr S_\ell$, it follows that G permutes transitively the ℓ entries of points of Ω. It then follows using Lemma 3.1 (b) that these 2ℓ partitions are permuted transitively by G. It also follows that N_α has exactly 2ℓ orbits of length 2 in Ω and these are permuted transitively by G_α.

We claim that, for two parts Σ, Σ' belonging to (not necessarily the same one of) these partitions, the intersection $\Sigma \cap \Sigma'$ cannot have size 2. Suppose to the contrary that $\Sigma = (\Sigma \cap \Sigma') \cup \{\beta\}$ and $\Sigma' = (\Sigma \cap \Sigma') \cup \{\gamma\}$ with $\beta, \gamma \in \Omega$, $\beta \neq \gamma$. Then N_β fixes Σ setwise, and hence N_β fixes $\Sigma \cap \Sigma'$ setwise. Therefore N_β also fixes Σ' setwise, so N_β fixes γ. Since N_β fixes a unique point in Ω this is a contradiction. Thus the claim is proved.

Finally we prove that $|\mathcal{D}| = 2\ell n$. Suppose that $\beta \in \Omega$, that Δ' is an N_β-orbit of length 2, and that $\Delta = \Delta'$. By the argument in the second paragraph of the proof, $\{\alpha\} \cup \Delta$ and $\{\beta\} \cup \Delta'$ are both parts of N-invariant partitions of Ω. Since they have at least two common points it follows from the previous paragraph that they must be equal, and hence $\alpha = \beta$. It follows that, for distinct points α, β of Ω, the N_α-orbits of length 2 in Ω are pairwise distinct from the N_β-orbits of length 2 in Ω, and hence $|\mathcal{D}| = 2\ell n$. *qed*

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