Counting symmetric configurations v_3

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Abstract

In this article we give tables of configurations v_3 for $v \leq 18$ and triangle-free configurations for $v \leq 21$ together with some statistics about some properties of the structures like transitivity, self-duality or self-polarity.

1 Definitions and Terminology

An *incidence geometry* $(P, \mathcal{B})$ consists of a set of v *points* $P = \{p_1, \ldots, p_v\}$ and a collection of b *blocks* $\mathcal{B} = \{B_1, \ldots, B_b\}$ such that $B_i \subseteq P$ for $i = 1, \ldots, b$. The number of blocks containing a point $p \in P$ is called the *degree*, denoted $[p]$. Blocks are also called *lines* and $|B|$ is also called the *length* of the block B. A pair (p, B) with $p \in B \in \mathcal{B}$ is called a *flag*. In this case, the point p is said to lie on the line B. The line is said to *pass* through p. Additionally, in an incidence geometry any pair of points is joined by at most one line, i.e. $|B_i \cap B_j| \leq 1$ for each $i, j \in \{1, \ldots, b\}, i \neq j$.

An incidence geometry $(P, \mathcal{B})$ is called a *configuration* of type (v_r, b_k) if

1. $|B_j| = k$ for $j = 1, \ldots, b$
2. $[p_i] = r$ for $i = 1, \ldots, v$.

A configuration $(P, \mathcal{B})$ with $v = b$ is called *symmetric* (see, for instance [5]). Its type is simply denoted by v_r which is the same as b_k because $k = r$ in this case. The numbers of configurations v_3 for $v \leq 15$ are given in [1].

A configuration $C = (P, \mathcal{B})$ is called *decomposable* if it can be written as the union of two configurations C_1, C_2 on distinct point sets: $C_1 = (P_1, \mathcal{B}_1)$, $C_2 = (P_2, \mathcal{B}_2)$, $P = P_1 \cup P_2$ and $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2$. Indecomposable configurations are also called *connected*.

An *isomorphism* between two incidence geometries $C_1 = (P_1, \mathcal{B}_1)$ and $C_2 = (P_2, \mathcal{B}_2)$ is a bijective map $\alpha : P_1 \to P_2$ which maps $\mathcal{B}_1$ onto $\mathcal{B}_2$. Here, a block $B \in \mathcal{B}_1$ with $B = \{p_1, \ldots, p_k\}$ is mapped onto $B^\alpha := \{p_1^\alpha, \ldots, p_k^\alpha\}$. Thus, isomorphisms are incidence preserving maps, so that $p^\alpha \in B^\alpha$ if and only if $p \in B$. If $C_1 = C_2$, α is called an automorphism. The set of all automorphisms forms the group $\mathrm{Aut}(C)$.

Any subgroup $G \leq \mathrm{Aut}(C)$ may be seen as acting on points, blocks and flags. An orbit of G on points, blocks or flags is called a *point-*, *block-* or *flag-*, *orbit* (under G) respectively. If $\mathrm{Aut}(C)$ is transitive on P or $\mathcal{B}$ or the set of flags then C is said to be *point-*, *block-*, or *flag-transitive* respectively. If there is a cyclic subgroup of $\mathrm{Aut}(C)$ which acts transitively on the points, the configuration C is called *cyclic*. Clearly, flag transitivity implies both point- and block-transitivity.

A *blocking set* in C is a subset X of points such that each block contains at least one element of X and one element not belonging to X. A *blocking-set free configuration* contains no blocking sets.

A *triangle* of a configuration consists of three points, say a,b and c, such that the three pairs $\{a, b\}$, $\{b, c\}$, and $\{c, a\}$ are contained in different blocks. A configuration that has no triangles is called a *triangle-free configuration*.

Let $d(p) := \{B \in \mathcal{B} \mid p \in B\}$ and let $\mathcal{P} := \{d(p) \mid p \in P\}$. Clearly $C^d := (\mathcal{B}, \mathcal{P})$ is a configuration. It is called the *dual* of $(P, \mathcal{B})$. In the dual configuration we only reverse the role of points and blocks. Since d defines a natural bijection $P \to \mathcal{P}$ we can write $C^d = (\mathcal{B}, P)$ instead. Applying duality twice in a row we obtain a configuration that is isomorphic to the original one.

An isomorphism $C \to C^d$ is called an *anti-automorphism*. A configuration which admits an *anti-automorphism* is called *self-dual*. An anti-automorphism of order 2 is called a *polarity*. A configuration which admits a polarity is called *self-polar*.

Let C be a configuration, then $A(C)$ denotes the group of all its automorphisms and anti-automorphisms acting on $P \cup \mathcal{B}$. So $Aut(C)$ is a subgroup of $A(C)$. If C is self-dual it is of index 2, otherwise, $Aut(C) = A(C)$.

If the group $A(C)$ acts transitively on the flags (regarded as *unordered* pairs $\{P, \mathcal{B}\}$), C is called *weakly flag transitive*. Our computations did not give any configurations that are weakly flag transitive, but not flag transitive. It would be interesting to know whether such a configuration exists.

A good source of reference – especially for automorphism groups – is the famous book of Dembowski [4].

2 Results and Discussion

v	a	b	c	d	e	f	g	h	i
7	1	1	1	1	1	1	1	1	0
8	1	1	1	1	1	1	1	0	0
9	3	3	3	2	1	1	1	0	0
10	10	10	10	2	1	1	1	0	0
11	31	25	25	1	1	0	0	0	0
12	229	95	95	4	3	1	1	0	0
13	2,036	366	365	2	2	1	1	1	0
14	21,399	1,433	1,432	3	3	1	1	0	1
15	245,342	5,802	5,799	5	4	1	1	0	1
16	3,004,881	24,105	24,092	6	4	2	2	0	4
17	38,904,499	102,479	102,413	2	2	0	0	0	13
18	530,452,205	445,577	445,363	9	5	1	1	0	47

Table 1: Configurations (not necessarily connected)

v	a	b	c	d	e	f	g	h	i
15	1	1	1	1	0	1	1	0	0
16	0	0	0	0	0	0	0	0	0
17	1	1	1	0	0	0	0	0	0
18	4	2	2	0	0	0	0	0	0
19	14	6	6	0	0	0	0	0	0
20	162	40	40	1	0	1	1	0	0
21	4,713	307	303	1	0	0	0	0	0

Table 2: Triangle-free configurations

Here, a is the number of configurations v_3,
b is the number of self-dual configurations v_3,
c is the number of self-polar configurations v_3,
d is the number of point transitive configurations v_3,
e is the number of cyclic configurations v_3,
f is the number of flag transitive configurations v_3,
g is the number of weakly flag transitive configurations v_3,
h is the number of connected blocking-set free configurations v_3,
i is the number of disconnected (decomposable) configurations v_3.

In [6, 7] the problem of the existence of blocking-set free configurations is listed open for 8 cases. The results in Table 1 reduce the list of open cases to $v \in \{20, 23, 24, 26\}$.

Whereas for small v the automorphism group acts transitively on the points if and only if it acts transitively on the blocks, this does not hold in general. The exceptional cases found in our search were the configurations number 18.7 and 21.1 listed in the appendix and their duals.

Up to $v = 16$ the numbers in Table 1 have been computed independently by two different programs:

One written in Bayreuth constructs the objects as *regular linear spaces*: A linear space is an incidence structure with v points and b blocks such that each block contains at least two points and each pair of points is joined by a unique line. Each configuration can be turned into a linear space by adding lines of length two. Namely, for each pair of points which is not yet joined in the configuration we add an additional line of length two. Actually, what we get is a regular linear space in the sense of [1], that is: each point lies on the same number of lines for every length.

The other one was written in Bielefeld [2] and works with the so-called *Levi graph* of a configuration: The Levi graph of a configuration v_3 is the cubic bipartite graph with vertex set $P \cup \mathcal{B}$ with $p \in P$ and $B \in \mathcal{B}$ adjacent if and only if $p \in B$ (see [3]). The dual configuration gives the same Levi graph with the colour classes interchanged. It is easy to see that the Levi graph has girth at least 6. On the other hand every bipartite cubic graph with girth 6 is the Levi graph of one self-dual configuration or of two non self-dual configurations. The Levi graphs of the triangle-free configurations are exactly those with girth at least 8.

We think that an error occuring in both programs with the same effect is very unlikely. The numbers for $v > 16$ have not been checked independently so far.

In order to check the existence of blocking sets carefully, up to $v = 15$ not only the existence of blocking sets was checked by two independent programs, but in order to have a better check, these programs were even used to count the total number of blocking sets. They were in complete agreement.

Most of the Bielefeld jobs were run on a mixed cluster of alphas, suns, DECs, SGIs and mostly Linux Pentium PCs with 133MHz. The accumulated CPU needed for the generation of the structures and the computation of the properties listed in Table 1, normalized to Linux Pentium PCs with 133MHz, was e.g. 9.7 days for $v = 17$ and 141 days for $v = 18$.

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Appendix: Selected Configurations

The configurations are preceded by an identification number x.y with x the number of points and y a running number. The configurations are given by listing the blocks. So e.g. 012 034 056 ... denotes a configuration where the first block contains the points 0, 1 and 2, the second block contains the points 0,3 and 4, etc. Note that we substitute the numbers $10, 11, 12, \ldots$ by lower case letters a, b, $c, \ldots$. Cyclic configurations are denoted by listing a base block $\{i_1, i_2, i_3\}$. The blocks of the configuration can be obtained by forming all images of this block under the cyclic group $\langle(0, 1, \ldots, v-1)\rangle$. The order of the automorphism group is indicated in parentheses. All listed configurations are connected.

7.1: 013 mod 7 (168) cyclic (Projective plane of order 2, Heawood graph – Fano plane)

8.1: 016 mod 8 (48) cyclic (punctured affine plane of order 3, Möbius - Kantor configuration, Generalized Petersen graph GP(8,3))

9.1: 012 345 678 036 047 137 158 248 256 (12) intransitive with orbits of size 6 and 3 on points and blocks
9.2: 013 mod 9 (9) cyclic
9.3: 012 034 056 137 158 248 267 368 457 (108) flag transitive, not cyclic (Pappus configuration)

10.1: 012 036 147 258 348 357 456 079 189 269 (3)
10.2: 012 345 368 379 046 057 148 259 169 278 (2)
10.3: 013 mod 10 (10) cyclic
10.4: 014 235 067 189 268 379 058 157 249 346 (4)
10.5: 014 025 036 127 138 239 456 478 579 689 (24)
10.6: 023 145 067 089 168 179 246 257 349 358 (4)
10.7: 012 346 357 458 067 168 278 049 159 239 (3)
10.8: 012 034 056 137 158 247 268 359 469 789 (120) Desargues configuration, flag transitive
10.9: 012 034 056 789 137 158 247 268 369 459 (12)
10.10012 345 367 389 046 058 149 178 257 269 (6)
Apart from 10.3 and 10.8 all structures have intransitive automorphism groups.
Nevertheless, the sizes of the orbits on points and blocks are the same.

11.1: 013 mod 11 (11) cyclic

12.1: 013 mod 12 (12) cyclic
12.2: 018 mod 12 (24) cyclic
12.3: 089 mod 12 (12) cyclic
12.4: 012 034 056 137 189 25a 28b 3ab 468 49a 579 67b (72) flag transitive

13.1: 012 034 056 135 147 238 245 69a 6bc 79b 7ac 89c 8ab (96) blocking set free
13.2: 012 034 056 137 145 238 259 4ab 67c 68a 79b 8bc 9ac (2) self dual, not self polar
13.3: 013 mod 13 (13) cyclic
13.4: 01a mod 13 (39) cyclic, flag transitive

14.1: 012 034 056 137 148 239 25a 45b 689 6cd 7ac 7bd 8ad 9bc (2) self dual, not self polar
14.2: 013 mod 14 (14) cyclic
14.3: 019 mod 14 (14) cyclic

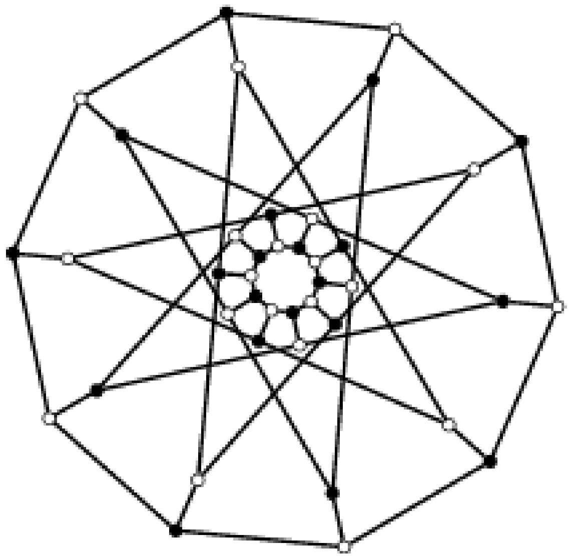

Figure 1: The Levi graph of the flag transitive triangle-free configuration on 20 points. It is the Kronecker double cover of the dodecahedron graph GP(10,2).

15.1: 012 034 056 135 147 238 269 46a 5bc 78b 79d 8ae 9ce acd bde (2) self dual, not self polar
15.2: 012 034 056 137 145 238 29a 4bc 5de 6bd 6ce 79e 7ad 89b 8ac (2) self dual, not self polar
15.3: 012 034 056 137 148 239 28a 4bc 59d 5ab 69e 6ac 7be 7cd 8de (2) self dual, not self polar
15.4: 013 mod 15 (15) cyclic
15.5: 01a mod 15 (15) cyclic
15.6: 015 mod 15 (30) cyclic
15.7: 01c mod 15 (30) cyclic
15.8: 012 034 056 178 19a 2bc 2de 37b 39d 48c 4ae 57e 59c 68d 6ab (720) flag transitive, triangle-free (Cremona-Richmond configuration, Tutte 8-cage)

16.1: 01a mod 16 (32) cyclic
16.2: 013 mod 16 (16) cyclic
16.3: 01c mod 16 (16) cyclic
16.4: 012 034 056 137 189 25a 28b 3ab 46c 4ad 5de 69e 79c 7bf 8ef cdf (32) point and block transitive
16.5: 012 034 056 137 189 25a 28b 3cd 46e 4bc 5df 69f 79e 7ad 8cf abe (96) flag transitive

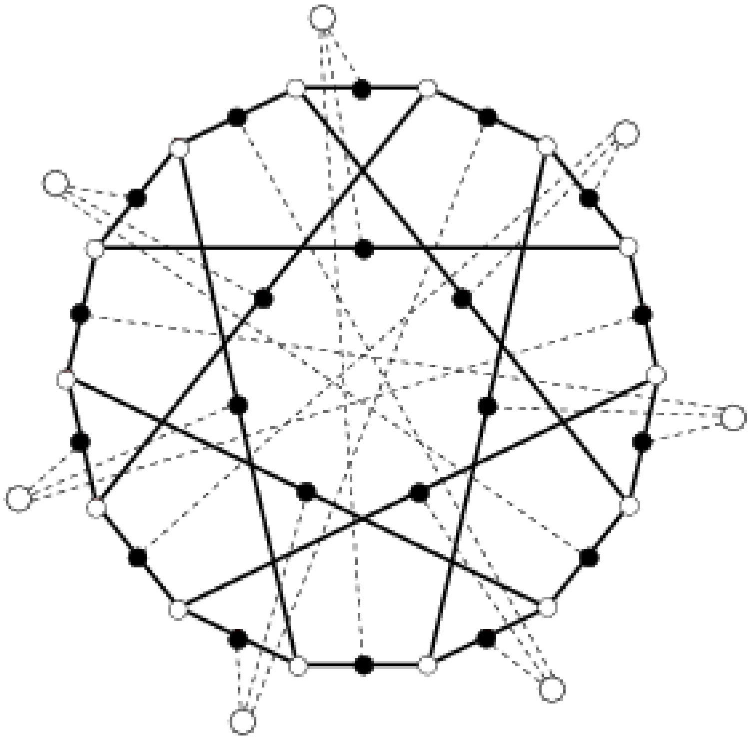

Figure 2: The Levi graph of the point transitive triangle-free configuration on 21 points can be obtained by subdividing the Heawood graph and connecting each set of 3 collinear vertices with valency 2 to a new vertex like sketched with dotted lines for one such set. The points of the configuration are the subdivision points of the Heawood graph.

17.1: 017 mod 17 (17) cyclic
17.2: 013 mod 17 (17) cyclic

18.1: 013 mod 18 (18) cyclic
18.2: 01b mod 18 (18) cyclic
18.3: 0ef mod 18 (18) cyclic
18.4: 01c mod 18 (18) cyclic
18.5: 012 034 056 137 189 2ab 2cd 3ac 48e 49f 57g 5be 67h 6df 8dg 9bh afg ceh (72) point and block transitive
18.6: 012 034 056 137 158 249 27a 368 47b 5cd 6ce 8de 9af 9bg abh cfg dfh egh (36) point and block transitive
18.7: 012 034 056 137 189 247 2ab 3cd 4ef 58a 5ce 69b 6df 7gh 8cg 9dh aeg bfh (144) point transitive, block intransitive (orbits of size 12 and 6)

20.1: 012 034 056 178 19a 2bc 2de 37f 3bg 49h 4di 58g 5ci 6af 6eh 7ch 8ej 9bj adg fij (240)
flag transitive, triangle-free (cf. Figure 1)

21.1: 012 034 056 178 19a 2bc 2de 37f 3bg 48h 49i 57d 5cj 6bi 6hk 8gj 9dk ach aeg efi fjk (42)
triangle-free, point transitive, block intransitive (orbits of size 13 and 7) (cf. Figure 2)