Tactical Decompositions and some Configurations v_4

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The study of configurations or – more generally – finite incidence geometries is best accomplished by taking into account also their automorphism groups. These groups act on the geometry and in particular on points, blocks, flags and even anti-flags. The orbits of these groups lead to tactical decompositions of the incidence matrices of the geometries or of related geometries. We describe the general procedure and use these decompositions to study symmetric configurations v_4 for small v. Tactical decompositions have also been used to construct all linear spaces on 12 points [2] and all proper linear spaces on 17 points [3] .

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1 Introduction

A *configuration* $\mathcal{C}$ of type (v_r, b_k) is an incidence geometry with v points $\mathcal{V} = \{p_1, \dots, p_v\}$ and a collection of b k-subsets of $\mathcal{V}$, called *blocks*: $\mathcal{B} = \{B_1, \dots, B_b\}$, $B_i \subseteq \mathcal{V}$ and $|B_i| = k$ for $i = 1, \dots, b$.

An incidence (or flag) is a pair $(p, B) \in \mathcal{V} \times \mathcal{B}$ with $p \in B$. Counting incidences in two ways gives $vr = kb$. A configuration with $v = b$ (and hence also $r = k$) is called *symmetric*. Symmetric configurations are denoted v_r for short.

An *isomorphism* φ between two geometries $\mathcal{C}_1 = (P_1, \mathcal{B}_1)$ and $\mathcal{C}_2 = (P_2, \mathcal{B}_2)$ is an incidence preserving map between the point sets. Let $\mathcal{V}_1 = \{p_1^{(1)}, \dots, p_k^{(1)}\}$ and $\mathcal{V}_2 = \{p_1^{(2)}, \dots, p_k^{(2)}\}$. So, for each block $B \in \mathcal{B}_1$ with $B = \{p_{i_1}^{(1)}, \dots, p_{i_k}^{(1)}\}$, $B^\varphi = \{p_{i_1}^{(1)^\varphi}, \dots, p_{i_k}^{(1)^\varphi}\}$ is a block of $\mathcal{B}_2$. An isomorphism of a geometry with itself is an *automorphism* (or *collineation*). The set of automorphisms forms a group with respect to composition of mappings, denoted by $\mathrm{Aut}(\mathcal{C})$.

An incidence geometry $\mathcal{C} = (\mathcal{V}, \mathcal{B})$ gives rise to a 0/1-matrix, the *incidence matrix*: Put $N = (n_{i,j})$ with $n_{i,j} = 1$ or 0 whether or not p_i is contained in B_j. Relabelling of points results in a permutation of the rows of this matrix, rearranging the order of the bocks yields a permutation of the columns of this matrix. Thus, the incidence matrix is unique up to reordering of rows and columns.

In this note we often draw incidence matrices by rectangular grids using little boxes to indicate incidences (see below). However, we stick to the original definition of a 0/1-matrix when talking about the row- or column-sums of N.

2 Tactical Decompositions

Let P be set. A set partition $P = (P_1, P_2, \ldots, P_r)$ is a sequence of subsets $P_i \subseteq P$ called *parts* (or classes) of P with $P_i \cap P_j = \emptyset$ for $i \neq j$ and $\bigcup_{i=1}^{r} P_i = X$. Usually, one allows rearranging the parts of a partition. Sometimes, the ordering of the parts is considered significant, in which case we call the partition *ordered*. The number r is called the *length* of the partition, denoted by $\ell(P)$. For P a partition and i a natural number let P_i denote the i-th part of P if $i \leq \ell(P)$ and the empty set otherwise. For P a partition let $\|P\| = (|P_1|, |P_2|, \ldots, |P_{\ell(P)}|)$ be the vector of class lengths.

A *decomposition* of an incidence geometry $\mathcal{C} = (\mathcal{V}, \mathcal{B})$ is a pair of set partitions of points and blocks. Let $(\Omega, \Delta) \in \Pi(\mathcal{V}) \times \Pi(\mathcal{B})$ be a decomposition of $\mathcal{C} = (\mathcal{V}, \mathcal{B})$. For $i \leq \ell(\Omega)$ and $j \leq \ell(\Delta)$ put

$$\alpha_{i,j} = |\{B \in \Delta_j \mid p \in B\}|$$

with $p \in \Omega_i$ fixed. In addition, put

$$\beta_{i,j} = |\{p \in \Omega_i \mid p \in B\}|$$

for fixed $B \in \Omega_j$. The decomposition (Ω, Δ) is called *row-tactical*, if for any $i \leq \ell(\Omega)$ and $j \leq \ell(\Delta)$ the number $\alpha_{i,j}$ is independent of the choice of $p \in \Omega_i$. The decomposition (Ω, Δ) is called *column-tactical*, if for any $i \leq \ell(\Omega)$ and $j \leq \ell(\Delta)$ the number $\beta_{i,j}$ is independent of the choice of $B \in \Delta_j$. The decomposition is called *tactical (TD)* if it is both, row- and column-tactical.

Any decomposition allows to reorder rows and columns of the incidence matrix in order to group together rows and columns according to the classes of the decompositions. Thus, any decomposition gives rise to a block decomposition of the incidence matrix N. The submatrices of size $|P_i| \times |B_j|$ are the *decomposition matrices*.

The matrices containing the $\alpha_{i,j}$ and the $\beta_{i,j}$ extended by one row and column indicating the order of the point and block classes are the *row* and *column decomposition schemes* (or TD-schemes). Let (Ω, Δ) be a decomposition and put $\ell(\Omega) = r$ and $\ell(\Delta) = s$. Then the row decomposition scheme and the column decomposition scheme have the following arrays of numbers

$$\begin{array}{c|cccc} & |\Delta_1| & |\Delta_2| & \ldots & |\Delta_s| \\ \hline |\Omega_1| & \alpha_{1,1} & \alpha_{1,2} & \ldots & \alpha_{1,s} \\ |\Omega_2| & \alpha_{2,1} & \alpha_{2,2} & \ldots & \alpha_{2,s} \\ \vdots & \vdots & & & \vdots \\ |\Omega_r| & \alpha_{r,1} & \alpha_{r,2} & \ldots & \alpha_{r,s} \end{array} , \quad \begin{array}{c|cccc} & |\Delta_1| & |\Delta_2| & \ldots & |\Delta_s| \\ \hline |\Omega_1| & \beta_{1,1} & \beta_{1,2} & \ldots & \beta_{1,s} \\ |\Omega_2| & \beta_{2,1} & \beta_{2,2} & \ldots & \beta_{2,s} \\ \vdots & \vdots & & & \vdots \\ |\Omega_r| & \beta_{r,1} & \beta_{r,2} & \ldots & \beta_{r,s} \end{array} .$$

For a tactical decomposition, the equation

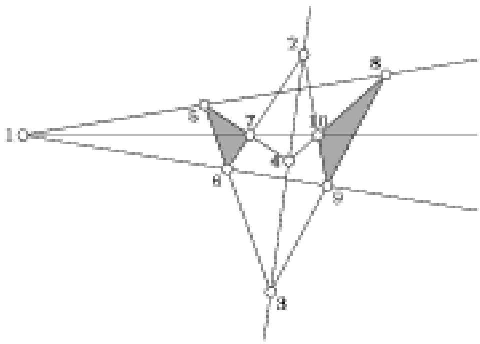
$$|\Omega_i| \cdot \alpha_{i,j} = \beta_{i,j} \cdot |\Delta_j| \tag{1}$$

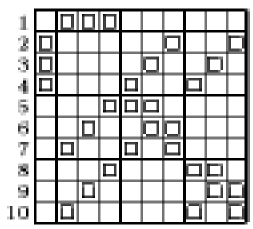
allows to switch between the row-tactical and the column-tactical scheme. This equation is proved by a simple double count of the set of incidences between points of class Ω_i and blocks of class Δ_j:

$$|\Omega_i| \cdot \alpha_{i,j} = |\{(p, B) \in \Omega_i \times \Delta_j \mid p \in B\}| = \beta_{i,j} \cdot |\Delta_j|.$$

We give some examples of tactical decompositions:

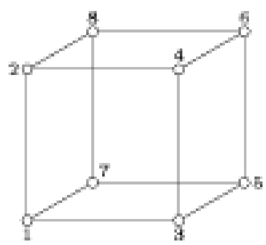
1. (The Desargues Configuration): In the Desargues Configuration we have two triangles in perspective position. In the figure these are indicated by a shadow. We also display the corresponding incidence matrix which is decomposed tactically. The tactical decomposition is described by its row decomposition scheme.

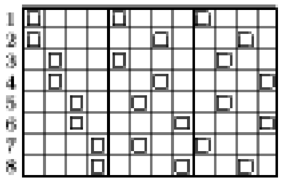




	1	3	3	3
1	0	3	0	0
3	1	0	1	1
3	0	1	2	0
3	0	1	0	2

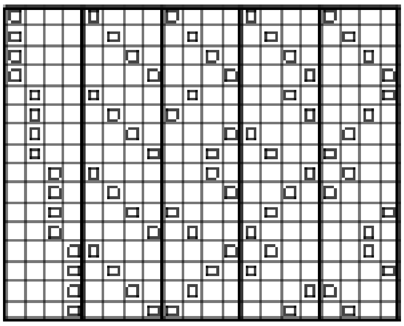
2. (The cube): The incidence matrix for the cube can be decomposed into three block-parts.





	4	4	4
8	1	1	1

3. (The affine plane) The affine plane of order 4 is a composition of five parallel classes. Its incidence matrix may therefore be decomposed in the following way:



	4	4	4	4	4
16	1	1	1	1	1

4. (Kirkman system on 15 points) As another example consider the seven days walk of Kirkman's schoolgirls. It may be indicated by the following TD-scheme:

	5	5	5	5	5	5	5
15	1	1	1	1	1	1	1

.

A very useful procedure for studying geometries comes from design theory. Therefore, assume the point p is fixed and consider the two sets of blocks

$$\mathrm{der}_p(\mathcal{C}) \;:=\; \{B\backslash p \mid B \in \mathcal{B},\, p \in B\}$$

and

$$\mathrm{res}_p(\mathcal{C}) \;:=\; \{B \in \mathcal{B} \mid p \notin B\}$$

called the *point-derivation* and the *point-residuum* (with respect to the point p). Often, der and res lead to tactical decompositions, for example for the $STS(13)$, $STS(15)$ and $STS(19)$ we get

	6	20
1	6	0
12	1	5

	7	28
1	7	0
14	1	6

	9	48
1	9	0
18	1	8

so we get configurations $12_5 20_3$, $14_6 28_3$ and $18_8 48_3$ as point residuals respectively. It is well known that there are exactly 5 configurations $12_5 20_3$ and 787 $14_6 28_3$ [1]. One can consider them as regular linear spaces of type $(12|7,20)$, $(14|7,28)$ and $(18|9,48)$, where the 2-lines intersect in an additional point. The number of configurations $18_8 48_3$ is unknown (it should be very large).

Another procedure is *block derivation*. Therefore, fix a block H and consider the sets

$$\mathrm{der}_H(C) \;:=\; \{B\backslash H \,|\, B \in \mathcal{B},\, B \cap H \neq \emptyset\}$$

and

$$\mathrm{res}_H(C) \;:=\; \{B \in \mathcal{B} \,|\, B \cap H = \emptyset\}.$$

They are the *block derived* and *residual geometries*. For the STS on 13, 15 and 19 points, we get

	1	15	10
3	1	5	0
10	0	3	3

	1	18	16
3	1	6	0
12	0	3	4

	1	24	32
3	1	8	0
16	0	3	6

Here, we find configurations 10_3, $12_4 16_3$ and $16_6 32_3$ as block residui. There are exactly 10 configurations 10_3 and 574 configurations of type $12_4 16_3$ (which are regular linear spaces of type $(10|15,10)$ and $(12|18,16)$ respectively).

Point derivation in Steiner Systems $S(2,4,13)$ (the unique projective plane of order 3, see below), $S(2,4,16)$ (the unique affine plane of order 4) and $S(2,4,25)$ (compare Spence [8]): gives

	4	9
1	4	0
12	1	3

	5	15
1	5	0
15	1	4

	8	42
1	8	0
24	1	7

Here, we get configurations $13_3 9_4$, 15_4 and $24_7 42_4$ as residual structures (all with a parallel class of lines of length 3). They can be seen as regular linear spaces of type $(12|0,4,9)$, $(15|0,5,15)$ and $(24|0,8,42)$ where the 3-lines intersect in an additional point. The configuration 15_4 occurring here is the affine plane of order 4 with one parallel class removed. There are three other configurations 15_4 as we will see in the sequel.

Block derivation leads to the following TD-schemes:

	1	12
4	1	3
9	0	4

	1	16	3
4	1	4	0
12	0	4	1

	1	28	21
4	1	7	0
21	0	4	4

The configurations 21_4 occur as residual structure in the last case. In the first case, we get the affine plane of order 3 as a derived structure. The middle one has configurations $12_4 16_3$ as derived structures.

There also exist tactical decompositions for flag-derived structures. For the $S(2,4,13)$, $S(2,4,16)$ and $S(2,4,25)$ we get

	1	3	9
1	1	3	0
3	1	0	3
9	0	1	3

	1	4	12	3
1	1	4	0	0
3	1	0	4	0
12	0	1	3	1

	1	7	21	21
1	1	7	0	0
3	1	0	7	0
21	0	1	3	4

Here, we find configurations 9_3, 12_3, 21_3 and 21_4.

3 The TDA and the Structure of Geometries

An important way to obtain tactical decompositions of an incidence system $\mathcal{D} = (\mathcal{V}, \mathcal{B})$ is by considering the orbits of a group of automorphisms A of $\mathcal{S}$. This means that we start out with a group $A \leq \mathrm{Aut}(\mathcal{S})$ and consider the two set partitions

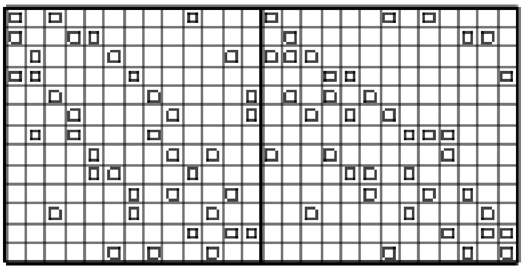
$$\Omega = \mathcal{V}//A \quad \text{and} \quad \Delta = \mathcal{B}//A,$$

yielding the decomposition (Ω, Δ) which we call A-decomposition (here, $X//A$ means the set of orbits of the group A on the set X). The decomposition induced by the full automorphism group, the $\mathrm{Aut}(\mathcal{S})$-decomposition, is called *TDA* (*tactical decomposition by automorphisms*).

3.1 Lemma *Let $\mathcal{S} = (\mathcal{V}, \mathcal{B})$ be an incidence structure. Let $A \leq \mathrm{Aut}(\mathcal{S})$ be a group of automorphisms. Then the A-decomposition is tactical.*

Proof: Let $p \in \Omega_i$ and $B \in \Delta_j$ be a point/block pair from the i-th point and the j-th block class. Then for any $a \in A$, $p^a \in P_i$ and $B^a \in B_j$ and $p \in B \iff p^a \in B^a$ holds. From the fact that a induces bijections on Ω_i and Δ_j we deduce that the number of incidences in each row is a constant, $\alpha_{i,j}$, and the number of incidences in each column is a constant, $\beta_{i,j}$. □

We consider as an example the decomposition by automorphisms of the two Steiner systems on 13 points (compare also Mathon, Phelps, Rosa [6]). One is cyclic and has an automorphism group of order 39 (cf. Figure 1). The other one has an automorphism group of order 6. This group leads to a rather fine TDA-decomposition (cf. Figure 2).

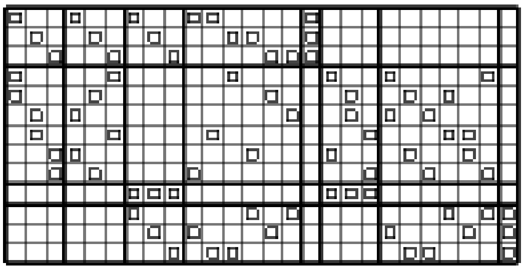


	13	13
13	3	3

Figure 1: TDA decomposition of the cyclic STS(13)

One decomposition is always possible, it is called the *trivial* one and consists of the *discrete decomposition* where each point and each block form a single part in Ω and Δ respectively. This decomposition is in fact the A-decomposition in the above sense if the automorphism group A is trivial. The matrices $(\alpha_{i,j})$ and $(\beta_{i,j})$ from the row-tactical and column-tactical decomposition scheme coincide with the incidence matrix of the geometry.

There are other decompositions which do not necessarily come from orbits of certain automorphism (sub)groups. For example, each configuration admits another trivial decomposition which has only one decomposition matrix, namely the whole incidence matrix. This is the decomposition with $r = s = 1$ and $\Omega_1 = \mathcal{V}$ and $\Delta_1 = \mathcal{B}$. Here, $\alpha_{1,1} = r$ and $\beta_{1,1} = k$.



	3	3	3	6	1	3	6	1
3	1	1	1	2	1	0	0	0
6	1	1	0	1	0	1	2	0
1	0	0	3	0	0	3	0	0
3	0	0	1	2	0	0	2	1

Figure 2: TDA decomposition of second STS(13)

An important notion for decompositions is the following: A decomposition (Ω, Δ) is called *invariant* if the blocks of Ω and Δ are either mapped onto itself or onto different blocks of the same decomposition. This means that the sets of blocks of the decomposition are permuted among themselves. Looking at the automorphism group as a permutation group on the points this means that Ω is a *system of imprimitivity* for $\mathrm{Aut}(C)$. The subgroup which maps each block onto itself is normal and will be called the *kernel* (of the decomposition). If the kernel coincides with the full automorphism group the decomposition is called *characteristic*. Finally we call an incidence matrix *indecomposable* if it contains no non-trivial TD. We call the incidence matrix *primitive* if it admits no non-trivial invariant TD. We remark that one would like to classify all primitive configurations and build up each incidence matrix by primitive configurations.

Examples:

1. The decomposition of the Desargues configuration in the example above is *not* invariant: The automorphism group is transitive on points but the decomposition contains one isolated point.

2. The TD for the cube is invariant since the three subrectangles corresponding to the three dimensions of space must be permuted. Here, the kernel is the group $Z_2 \times Z_2 \times Z_2$. The full collineation group is an extension of the kernel by the symmetric group S_3 on the set of three dimensions. It is isomorphic to the wreath product $S_2 \wr S_3$.

3. The decomposition of the affine plane of order 4 into 5 parallel classes has as kernel the dilatation group of the plane. It has order 48. The full group of automorphisms has order 5760.

4. An example for a characteristic TD is the TD of a Latin square, say of order 5.

	3	25
15	1	5

Since each collineation permutes the three 5-blocks and permutes the 25 3-blocks, it fixes both subrectangles.

4 Symmetric Configurations v_4

A symmetric configuration v_4 consists of n points and v lines such that each line has length 4 and each point is incident with 4 lines. Furthermore, each pair of points is incident with at most one line. Thus, we have a TD-scheme

$$\begin{array}{c|c|} & v \\ \hline v & 4 \end{array}.$$

We enumerate such configurations for small v:

Result: The numbers $N(v_4)$ of configurations v_4 for small v are the following:

v :	13	14	15	16	17	18
$N(v_4)$:	1	1	4	19	1,972	971,171

The automorphism groups have the following orders or distributions of orders:

$$\begin{array}{ll} v = 13: & 5616; \\ v = 14: & 336; \\ v = 15: & 15, 30, 24, 360; \\ v = 16: & 1^2, 2^6, 3^3, 4^1, 6^2, 12^1, 16^1, 18^1, 32^1, 1152^1; \\ v = 17: & 1^{1761}, 2^{134}, 3^{24}, 4^{27}, 6^{13}, 8^5, 12^2, 17^2, 18^1, 36^2, 72^1. \end{array}$$

We will give descriptions of some of these configurations in the sequel. For $v = 13$ we have the well known projective plane of order 3. Using a Singer cycle, we may give a *cyclic representation* of this geometry: Take the point set $\{0, 1, ..., 12\}$ and define the following blocks: $\{0, 1, 4, 6\}^{C_{13}}$.

5 The Symmetric Configuration 14_4:

The unique symmetric configuration 14_4 has the following property: for each point p there is exactly one point p' not joined with p. So we have a natural partition of the 14 points into 7 pairs of points. Dually also the 14 blocks are divided into 7 pairs. The related TD is shown in Figure 3. Turning now to the quotient scheme of 49 elements we see that the empty blocks of this quotient structure form the incidence matrix of the Fano (= 7-point) plane.

Another description of the configuration 14_4 results from the TD-scheme

$$\begin{array}{c|cc} & 7 & 7 \\ \hline 7 & 3 & 1 \\ 7 & 1 & 3 \end{array}$$

and is shown in Figure 4. Here two Fano planes are interlaced: each block of the first

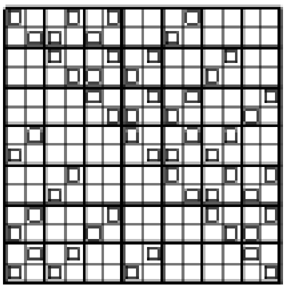


Figure 3: The configuration 14_4

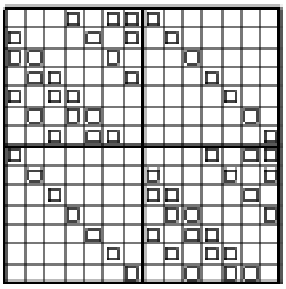


Figure 4: The configuration 14_4 as two Fano planes

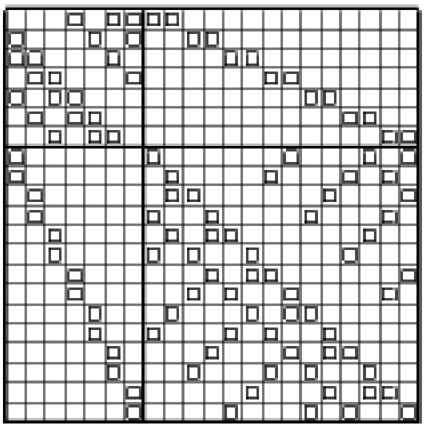
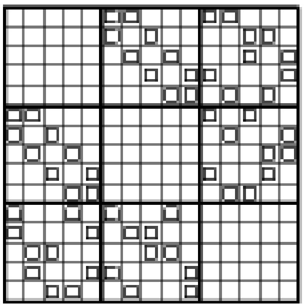


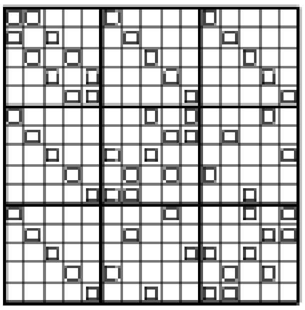
Figure 5: The configuration 14_4 as complement of the Baer subplane in $PG_2(4)$

one is extended by a point of the second one to form a 4-block and vice versa. The best understanding of the situation may be the following (Dembowski [4] p. 305 or Gropp [5]): Take the projective plane over the field $GF(2^2)$ and in it the Baer subplane corresponding to the subfield $GF(2)$. Then the configuration 14_4 turns out to be the complement of the Baer subplane in the whole plane, see Figure 5. The collineation group of 14_4 is now generated by the collineation group of the Baer subplane and the Baer involution. Such a geometry is called an *elliptic semiplane* since for two distinct points p and q there is at most one line through p and q and we have the following generalization and dualization of the axiom of parallels in an affine plane: For any non-incident point line pair $p \notin B$ there is at most one line through this point p parallel to B and there is at most one point q on B such that $[p,q] = \emptyset$. It is called elliptic because the degrees of elements are all $v+1$.

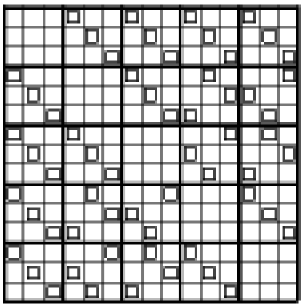
We note that the configuration 14_4 may also be cyclically generated: $\{0,2,3,7\}^{C_{14}}$.



	5	5	5
5	0	2	2
5	2	0	2
5	2	2	0



	5	5	5
5	2	1	1
5	1	2	1
5	1	1	2



	3	3	3	3	3
3	0	1	1	1	1
3	1	0	1	1	1
3	1	1	0	1	1
3	1	1	1	0	1
3	1	1	1	1	0

Figure 6: The cyclic configurations 15_4

6 The Four Configurations 15_4

Three of them are cyclically generated:

$$\begin{array}{ll} \{0,2,8,12\}^{C_{15}}: & |Aut| = 15, \\ \{0,1,9,11\}^{C_{15}}: & |Aut| = 30, \\ \{0,1,9,13\}^{C_{15}}: & |Aut| = 360. \end{array}$$

Possible TDs for these three configurations are shown in Figure 6.

The extension of the kernel K by Aut/K is in the first case $C_5 \rtimes C_3$; in the second case it is $C_5 \rtimes S_3$ where S_3 acts as symmetric group on the three squares on the main diagonal, and in the third case we have $C_3 \rtimes S_5$. Here the symmetric group S_5 acts on the five empty diagonal fields. The third configuration has the following interpretation (compare Section 2): it comes from the projective plane of order 4 by removing a non-incident point-line-pair $p \notin L$ and all points on L and all lines through p. The kernel C_3 of the TD is in this interpretation the group of homologies with respect to the antiflag $p \notin L$, and it is extended by the group $S_5 \cong P\Gamma L(2,4)$ acting on the line L.

	3	12
3	0	4
12	1	3

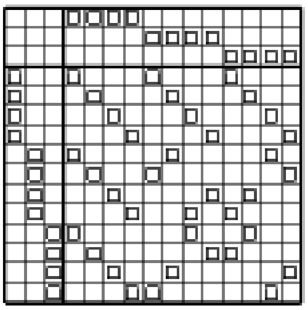


Figure 7: The fourth configuration 15_4 decomposed characteristically

Figure 7 shows a characteristic TD together with the incidence matrix for the remaining fourth configuration 15_4 with $|Aut| = 24$. This configuration is a substructure of the projective plane of order 5: Choose in the projective plane PG(2,5) a non-incident point-line-pair, say the origin and the line L_∞. Now remove the following 16 points: the origin $(0,0)$, the 12 points $(x,0),(0,y),(x,x), x,y = 1,2,3,4 \in GF(5)$ and the three points $(2),(3),(4) \in L_\infty$. Dually we remove the following 16 lines: the line L_∞, the 16 lines through the points $(2),(3),(4)$ different from L_∞ and not incident with the origin, and the three lines which join the origin and the points $(0),(1),(\infty)$.

Note that the subconfiguration 12_3 also has an automorphism group of order 24; it is a subgeometry of the affine plane of order 5.

Historically, three of these configurations have already been known to Merlin [7].

7 The 19 Configurations 16_4:

Since these geometries might not be too well known, let us give a complete list of them. We denote the 16 blocks by numbers and letters $0,1,2,3,4,5,6,7,8,9,a,b,c,d,e,f$. We list the incidence matrices in a row-by-row fashion, specifying the set of blocks incident to a given point as a 4-tuple. An empty space in the following table means that the 4-tuple coincides with the corresponding 4-tuple of the preceeding configuration one row above. The first four rows coincide for all geometries. They are

$$0123 \quad 0456 \quad 0789 \quad 0abc \; .$$

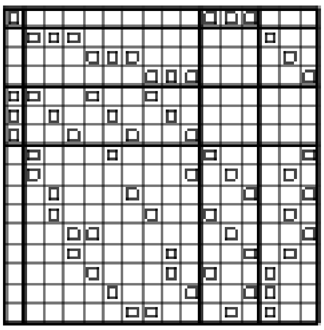
The remaining 12 rows of all 19 configurations are shown in Table 1 (the order of the automorphism group is indicated in parentheses). Let us first look for species which are cyclically generated. There are two configurations whose automorphism groups have order 16 or 32. Both are indeed cyclically generated:

$$\begin{array}{lll} 11) & \{0,1,4,6\}^{C_{16}}: & |Aut| = 16; \\ 13) & \{0,1,6,13\}^{C_{16}}: & |Aut| = 32. \end{array}$$

The configuration with automorphism group order 1152 is the affine plane of order 4 where one parallel class is removed. We are interested in those geometries which have a rather large group. Here we notice that the three geometries with group order $12, 18, 1152$ all admit the following TD:

	4	12
4	1	3
12	1	3

As an example we show the configuration with Aut = 18, (no 18 in the list), see Figure 8.



$$\begin{array}{rcl} |Aut| &=& 18 \\ Aut &=& \langle (2\,3)(6\,7)(8\,12)(9\,14)(11\,16)(13\,15), \\ && (2\,4\,3)(8\,11\,14)(9\,16\,12)(10\,13\,15), \\ && (5\,6\,7)(8\,10\,12)(9\,11\,13)(14\,15\,16) \rangle \end{array}$$

Figure 8: A configuration 16_4

1)147a,158b,169c,248d,257e,26af,349f,35cd,36be, 7bdf, 8cef, 9ade (2)

2) − −, − −, − −, − −, − −, − −,359f,36bd, 38ce, 4bef,7cdf, 9ade (1)

3) − −, − −, − −, − −,259e,267f,34be, 35cf, 36ad,7cde,8aef,9bdf (3)

4) − −, − −, − −, − −, − −, − −,34cf,35ad,36be, 7cde,8aef,9bdf (6)

5) − −, − −,169d,248c,259e,267f,35ad,36be, 39cf, 4bdf, 7cde, 8aef (2)

6) − −, − −, − −,248e,259c,267f,34bd,36ae,38cf,5adf,7cde, 9bef (1)

7) − −, − −, − −, − −, − −, − −,36ce, 37bd, 38af,4cdf, 5ade,9bef (2)

8) − −, − −, − −, − −, − −,26bf, 357f,36ce, 38ad,4cdf, 7bde, 9aef (12)

9) − −, − −, − −, − −,25cd,26af,349b,38cf, 3ade,59ef,67ce, 7bdf (3)

10) − −, − −, − −,249b, 25ce, 2adf, 359f,36ae,38cd, 48ef,67cf, 7bde (2)

11) − −, − −, − −, − −,25de,26cf, 359f,36ae,37cd, 48ce, 7bef, 8adf (16)

12) − −, − −, − −, − −,26ce, 2adf, 35ae, 367f,38cd, 48ef,59cf, 7bde (4)

13) − −, − −, − −,249c, 25de,268f,34be, 35cf, 38ad,67ce, 7bdf, 9aef (32)

14) − −, − −,16de,248d,25ae,269c, 37cd, 38af, 39be, 4cef, 59df, 67bf (6)

15) − −, − −, − −, − −, − −, − −,37ce, 38af,39bd, 49ef,5cdf, 67bf (2)

16) − −, − −, − −, − −, − −,269f,367c,38af,39bd,49ce,5cdf, 7bef (3)

17) − −, − −, − −, − −,269c,2bef, 34cf,37bd, 39ae,57ce,59df, 68af (2)

18) − −, − −,1def,249d,26ae,27cf, 35cd, 368f,39be, 48ce, 59af,67bd (18)

19)147d,15ae,18bf,24cf,268e,29ad,359f,36bd,37ce, 49be, 58cd,67af (1152)

Table 1: The configurations 16_4

8 The Configurations 17_4:

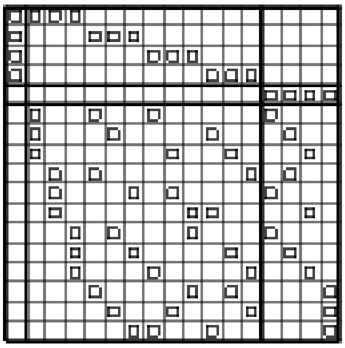
There are 1972 configurations 17_4. Those 26 with an automorphism group of order ≥ 5 are shown in the following table. They all have the first 4 rows equal to

$$0123, 0456, 0789, 0abc$$

so we list only the following 13 rows (cf. Table 2). There are only two geometries in the list whose automorphism group order is a multiple of 17, and these two geometries are cyclically generated:

$$\{0, 1, 4, 6\}^{C_{17}}: \quad |Aut| = 17;$$
$$\{0, 1, 5, 15\}^{C_{17}}: \quad |Aut| = 17.$$

It is a task of its own to study and analyze all these 1972 geometries. As an example we only present the configuration with the largest collineation group (of order 72), see Figure 9. Note: If we remove the distinguished point and the distinguished block and add 4 diagonal incidences in the upper right quadrangle, then we arrive at a related configuration 16_4.



$$
\begin{aligned}
|Aut| &= 72 \\
Aut &= \langle (1\,2)(7\,9)(8\,15)(10\,12)(11\,16)(14\,17), \\
&\quad (1\,3\,2)(7\,14\,15)(8\,17\,9)(11\,16\,13), \\
&\quad (1\,4\,3\,2)(6\,7\,11\,15)(8\,17\,12\,9)(10\,13\,14\,16)\rangle
\end{aligned}
$$

Figure 9: A configuration 17_4

42) 147*a*, 158*b*, 169*d*, 248*c*, 257*e*, 2*adf*, 349*g*, 368*f*, 3*bde*, 5*cdg*, 6*aeg*, 7*bfg*, 9*cef* (6)
431) −−, −−, −−, 248*e*, 25*cd*, 27*fg*, 357*e*, 36*cf*, 3*bdg*, 49*bf*, 68*ag*, 9*ceg*, *adef* (8)
433) −−, −−, −−, −−, −−, −−, −−, 39*cf*, 3*adg*, 49*bg*, 68*af*, 6*ceg*, *bdef* (8)
614) −−, −−, −−, 249*b*, 25*ce*, 2*afg*, 359*f*, 36*cg*, 3*ade*, 48*dg*, 68*ef*, 7*beg*, 7*cdf* (6)
617) −−, −−, −−, −−, 25*de*, 26*fg*, 359*f*, 36*ce*, 3*adg*, 48*cg*, 7*beg*, 7*cdf*, 8*aef* (17)
623) −−, −−, −−, −−, 25*ef*, 2*cdg*, 359*g*, 37*ce*, 3*adf*, 48*de*, 68*cf*, 6*aeg*, 7*bfg* (6)
632) −−, −−, −−, −−, 26*ef*, 28*dg*, 36*cg*, 37*be*, 3*adf*, 48*cf*, 57*fg*, 5*cde*, 9*aeg* (6)
1298) −−, −−, 16*de*, 248*d*, 257*e*, 26*fg*, 369*c*, 37*bf*, 38*ag*, 4*ceg*, 5*cdf*, 9*aef*, 9*bdg* (6)
1315) −−, −−, −−, −−, 257*f*, 29*be*, 35*cd*, 368*g*, 3*aef*, 4*bfg*, 69*cf*, 7*ceg*, 9*adg* (6)
1614) −−, −−, −−, −−, 27*bf*, 29*ae*, 35*ce*, 36*af*, 39*bg*, 4*efg*, 57*dg*, 68*cg*, 9*cdf* (17)
1744) −−, −−, −−, 248*f*, 27*cd*, 29*ae*, 359*d*, 367*g*, 3*bef*, 4*bdg*, 5*afg*, 69*cf*, 8*ceg* (6)
1854) −−, −−, −−, 249*d*, 26*bf*, 27*cg*, 34*ce*, 368*g*, 39*af*, 57*ef*, 5*adg*, 8*cdf*, 9*beg* (8)
1916) −−, −−, 1*def*, 249*d*, 267*e*, 2*afg*, 35*cd*, 36*bf*, 38*eg*, 48*cf*, 59*ae*, 69*cg*, 7*bdg* (6)
1917) −−, −−, −−, −−, 26*ae*, 27*cf*, 359*e*, 36*bf*, 38*cd*, 48*fg*, 5*adg*, 69*cg*, 7*beg* (6)
1918) −−, −−, −−, −−, −−, −−, 35*cd*, 368*f*, 39*be*, 48*eg*, 5*afg*, 69*cg*, 7*bdg* (36)
1919) −−, −−, −−, −−, −−, −−, −−, −−, −−, 4*bfg*, 57*eg*, 69*cg*, 8*adg* (18)
1945) −−, 158*d*, 16*be*, 249*d*, 25*bf*, 28*cg*, 34*ce*, 368*f*, 39*bg*, 57*eg*, 6*adg*, 7*cdf*, 9*aef* (8)
1949) −−, −−, −−, −−, 268*c*, 2*aef*, 37*de*, 38*bf*, 39*ag*, 4*ceg*, 57*bg*, 59*cf*, 6*dfg* (12)
1952) −−, −−, −−, 249*e*, 25*bf*, 27*cg*, 34*cd*, 368*g*, 39*af*, 5*aeg*, 67*df*, 8*cef*, 9*bdg* (12)
1959) −−, −−, −−, −−, 25*cf*, 27*bd*, 34*dg*, 369*c*, 38*bf*, 59*ag*, 67*fg*, 8*ceg*, *adef* (6)
1962) −−, −−, −−, −−, 268*f*, 27*bg*, 34*cd*, 359*g*, 3*aef*, 57*cf*, 6*adg*, 8*ceg*, 9*bdf* (6)
1964) −−, −−, −−, 249*f*, 25*ce*, 28*ag*, 34*eg*, 37*bf*, 39*cd*, 59*bg*, 67*dg*, 68*cf*, *adef* (6)
1968) −−, −−, −−, −−, 25*eg*, 27*bd*, 34*cg*, 36*df*, 38*ae*, 57*cf*, 69*ag*, 8*bfg*, 9*cde* (8)
1970) −−, −−, −−, 259*f*, 267*g*, 2*cde*, 37*ef*, 38*bg*, 39*ad*, 49*cg*, 4*bdf*, 5*aeg*, 68*cf* (6)
1971) −−, −−, 16*ef*, 24*be*, 259*f*, 27*cd*, 369*b*, 38*cf*, 3*ade*, 49*dg*, 5*ceg*, 68*ag*, 7*bfg* (36)
1972) 147*d*, 158*e*, 1*afg*, 249*f*, 25*bg*, 28*cd*, 36*dg*, 37*cf*, 39*be*, 4*ceg*, 59*ad*, 67*ae*, 68*bf* (72)

Table 2: The configurations 17_4

References

[1] A. Betten and D. Betten, Regular linear spaces, *Beiträge Algebra Geom.* **38**, (1997), 111–124.

[2] A. Betten and D. Betten, Linear spaces with at most 12 points, *J. Comb. Des.* **7**, (1999), 119–145.

[3] A. Betten and D. Betten, The proper linear spaces on 17 points, *Discrete Appl. Math.*

[4] P. Dembowski, Finite geometries, *Classics in Mathematics*, Springer-Verlag, Berlin, 1997. Reprint of the 1968 original.

[5] H. Gropp, Nonsymmetric configurations with deficiencies 1 and 2, In *Combinatorics '90 (Gaeta, 1990)*, volume **52** of *Ann. Discrete Math.*, 227–239, Amsterdam, 1992, North-Holland.

[6] R. A. Mathon, K. T. Phelps, and A. Rosa, Small Steiner triple systems and their properties, *Ars Combin.* **15**, (1983), 3–110.

[7] E. Merlin, Sur les configurations planes n_4, *Academie Royale de Belgique, Bull. de la Classe des Sciences*, (1913), 647–660.

[8] E. Spence, The complete classification of Steiner systems $S(2,4,25)$, *J. Comb. Des.* **4**, (1996), 295–300.

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