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A Steiner 5-Design on 36 Points

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Dedicated to the memory of E. F. Assmus, Jr.

Abstract. Up to now, all known Steiner 5-designs are on $q+1$ points where $q \equiv 3 \pmod 4$ is a prime power and the design is admitting $PSL(2,q)$ as a group of automorphisms. In this article we present a 5-(36,6,1) design admitting $PGL(2,17) \times C_2$ as a group of automorphisms. The design is unique with this automorphism group and even for the commutator group $PSL(2,17) \times Id_2$ of this automorphism group there exists no further design with these parameters. We present the incidence matrix of t-orbits and block orbits.

Keywords: t-design, Steiner system, Kramer-Mesner method.

1. Introduction

For a long time, the only known t-designs had $t \leq 5$ admitting some group $PSL(2,q)$ as a group of automorphisms. The full automorphism group could be larger as in the case of the famous Witt designs [21]. Assmus and Mattson [2] contributed such designs for the cases $q = 23,\ 48$ deriving them from codes. The new designs had values of λ greater than 1, and in several cases consisted of just one orbit of the group $PSL(2,q)$. The search for t-designs at that time was closely related to the search for transitive extensions of permutation groups [14]. So, Assmus and Mattson pointed out that their new 5-designs are not "orbit-designs", i. e. $PSL(2,q)$ is not transitive on the set of blocks of the design. A few years later, Denniston [10, 9], Mills [18], Grannell, Griggs [11], constructed Steiner 5-systems, i. e. 5-$(q+1,k,1)$ designs, again using some $PSL(2,q)$ as a prescribed group of automorphisms. Denniston noticed that several orbits of the group on k-subsets had to be combined to obtain such a design. More generally, using the classification theorem of finite simple groups, Praeger and Cameron [7] showed that no block-transitive 8-(v,k,λ) design exists and conjectured that even no block-transitive 6-(v,k,λ) design does exist.

In order to construct t-designs as a combination of various orbits, Kramer and Mesner introduced a famous method in [15]. Using this approach, the authors developed a software package DISCRETA which already led to the discovery of several new t-designs with large t, that is, $t \geq 5$ (cf. [20, 6, 5]).

A difficult task in finding new t-designs by a software tool like DISCRETA still is to predict a group of automorphisms of the designs. In [8], all Steiner 5-designs were on $q+1$ points with $q \equiv 3 \pmod 4$ and q a prime power. All these designs admit $PSL(2,q)$ as a group of automorphisms. Denniston [10], Grannell, Griggs, Mathon [12, 13], Schmalz [19], and Mathon [17] looked for further values of q to construct new Steiner 5-systems. In this article, we slightly modify the permutation presentation of $PSL(2,17)$ yielding an action on a set of v points, where $v-1$ is not a prime power. So, we present a simple 5-(36, 6, 1) design admitting $PGL(2,17) \times C_2$ as a group of automorphisms.

The design is unique with this automorphism group and even for the commutator group $PSL(2,17) \times Id_2$ there is no other design with these parameters. We describe the Steiner system as a collection of k-orbits of our given group and show the incidence matrix between t-orbits and the k-orbits of the design. DISCRETA shows that there exist 5-(36, 6, λ) designs for each value of λ with $1 \le \lambda \le 31$ (and this prescribed group of automorphisms).

The interested reader may consult our WWW-page listing many t-designs especially for $t \ge 6$ and allowing to download DISCRETA for various platforms:

http://mathe2.uni-bayreuth.de/betten/DESIGN/d1.html

2. The Group

The prescribed group of automorphisms G has order 9792 and is isomorphic to $PGL(2,17) \times C_2$. Its permutation representation is obtained by taking two copies $X_1 = \{1,\ldots,18\}$ and $X_2 = \{19,\ldots,36\}$ of the natural point set of $PGL(2,17)$ with the same action of this group on both sets. A permutation of order two interchanging the corresponding points is taken as an additional generator of the group. Thus we obtain the following presentation by generating permutations.

$G =$

{(1 19)(2 20)(3 21)(4 22)(5 23)(6 24)(7 25)(8 26)(9 27)(10 28)(11 29)(12 30)(13 31)(14 32)
(15 33)(16 34)(17 35)(18 36),
(3 5 11 12 15 7 17 13 18 16 10 9 6 14 4 8)(21 23 29 30 33 25 35 31 36 34 28 27 24 32 22 26),
(3 8 4 14 6 9 10 16 18 13 17 7 15 12 11 5)(21 26 22 32 24 27 28 34 36 31 35 25 33 30 29 23),
(2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18)(20 21 22 23 24 25 26 27 28 29 30 31 32 33 34 35 36),
(1 3 11 8 15 9 5 7 17 4 14 16 12 6 13 10 18)(19 21 29 26 33 27 23 25 35 22 32 34 30 24 31 28 36)}

Note that this kind of direct product groups already led to other important t-designs, see Table 1.

We have first used $PGL(2,9) \times Id_2$ to find large sets. While in the case of an $LS[2](t,k,v)$ the value of λ in the t-designs is as large as possible up to complementary designs, in this article we are interested in the smallest possible value $\lambda = 1$. We have obtained some cases where $\lambda = 2$ or λ is the least possible value $\Delta\lambda$. So far the above Steiner system is the only one we could derive from this kind of groups. In the table, we list some parameter sets of designs stemming from various kinds of direct product groups. Some of these parameter sets are new, whilst others have already been found using groups of different type. In this table, $(M_{11})_{12}$ denotes the permutation representation of the Mathieu group M_{11} on 12 points.

Table 1. t-Designs from Direct Product Groups

$\Delta\lambda$	Parameter	Group	Remark
4	4-(12,5,4)	$PGL(2,5) \times Id_2$	$LS[2](4,5,12)$, [9]
1	3-(14,4,1)	$Hol(C_7)_2 \times Id_2$	[3]
6	3-(15,5,6)	$(Hol(C_7)_2 \times Id_2)+$	[4]
1	4-(15,5,3)	$(Hol(C_7)_2 \times Id_2)+$	[4]
2	3-(16,6,4)	$PSL(2,7) \times C_2$	[4]
1	5-(16,6,3)	$PGL(2,7) \times Id_2$	[4]
1	5-(16,6,5)	$PSL(2,7) \times Id_2$	[4]
7	4-(18,8,84)	$PGL(2,8) \times C_2$	[4]
168	4-(20,9,2184)	$PSL(2,9) \times Id_2$	$LS[2](4,9,20)$, [16]
28	4-(20,10,4004)	$PSL(2,9) \times Id_2$	$LS[2](4,10,20)$, [16]
70	5-(21,9,910)	$(PSL(2,9) \times Id_2)+$	$LS[2](5,9,21)$, [16]
280	6-(22,9,280)	$(PSL(2,9) \times Id_2)+$	$LS[2](6,9,22)$, [16]
1	5-(24,6,2)	$PSL(2,11) \times C_2$	
1	5-(24,8,288)	$(M_{11})_{12} \times Id_2$	
6	5-(24,9,1080)	$(M_{11})_{12} \times Id_2$	
1	5-(28,6,2)	$PGL(2,13) \times C_2$	
1	5-(34,6,5)	$P\Gamma L(2,16) \times C_2$	
1	5-(36,6,1)	$PGL(2,17) \times C_2$	
5	5-(40,6,5)	$PGL(2,19) \times C_2$	
20	5-(46,8,800)	$M_{23} \times Id_2$	
1	5-(48,6,2)	$PGL(2,23) \times C_2$	
1	5-(52,6,2)	$P\Gamma L(2,25) \times C_2$	
3	5-(56,6,3)	$P\Gamma L(2,27) \times C_2$	

3. Steiner Systems with Prescribed Automorphism Groups

In order to explain the choice of blocks yielding the 5-(36, 6, 1) design we make some remarks on the connection between the design and its automorphism group.

LEMMA 1 *Let G be a group of automorphisms of a t-$(v, k, 1)$ Steiner system $\mathcal{D} = (V, \mathcal{B})$. Then the mapping $\phi : \binom{V}{t} \to \mathcal{B}$ mapping each t-subset of the underlying point set V onto the unique block $B \in \mathcal{B}$ containing T commutes with the actions of G on $\binom{V}{t}$ and on $\mathcal{B}$. In particular, for each $T \in \binom{V}{t}$ its stabilizer $Stab_G(T)$ is contained in $Stab_G(\phi(T))$, the stabilizer of the block containing T. Any two t-subsets of a block B which are in the same orbit under G must be in the same orbit under $Stab_G(B)$.*

The 6-(36, 6, 1) design consists of 15 orbits of blocks under the action of $PGL(2,17) \times C_2$. It is easily veryfied that for each block-orbit representative B the sum over all $|\mathrm{Stab}_G(B)|/|\mathrm{Stab}_G(T)|$ for $T \subset B$ a representative of a 5-orbit is 6, as required by the lemma. The block orbits can be classified with respect to the orders of the stabilizers in G. The structure of the group action allows a refinement by the number of points belonging to the same orbit of $PGL(2,17)$.

So, each 5-subset T intersects the sets X_1 and X_2 in a certain number of points, which we call t_1 and t_2, respectively. Any $g \in PGL(2, 17) \subseteq G$ maps these sets of intersection only inside X_i and $h \in C_2 \subseteq G$ may exchange both sets. Therefore, we may take the set $\{t_1, t_2\}$ as an invariant of the orbit.

4. The 5-(36, 6, 1) Design

There are 48 orbits on 5-sets which are shown in the first column of Table 2. The Steiner system consists of 15 out of 259 orbits of G on 6-sets. A block of the Steiner system can be described as a 5-subset plus one additional point in the block. So, the first 15 rows of the table belong to 5-orbit representatives which are to be extended by one additional element to obtain the unique block of the design that contains that 5-set. We list the additional point in the third row at the top of the table. The remaining 5-orbits of G can be found in the following rows. The index appended to a 5-orbit representative denotes the order of the stabilizer of that 5-subset. The entry corresponding to the i-th orbit of 5-sets and the j-th block orbit counts the number of times an block of that particular orbit contains the fixed representative of the 5-orbit.

In Figure 1 we present graphs $A, \ldots, H$ showing the splitting of 6-orbit representatives into orbits on 5-sets. We get a bipartite graph with nodes corresponding to 5 and 6-orbits of the group, respectively. Here, we draw the 6-orbit at the top and the 5-orbits at the bottom. The labels at the nodes display the stabilizer order of each representative in G. Let K be a orbit representative of blocks of the design. The number of bonds reflect how many times elements of the 5-orbit appear as subsets of K (so, altogether 6 bonds). The t-orbit types are indicated at the bottom.

Figure 1. Splitting of Blocks into 5-Orbits

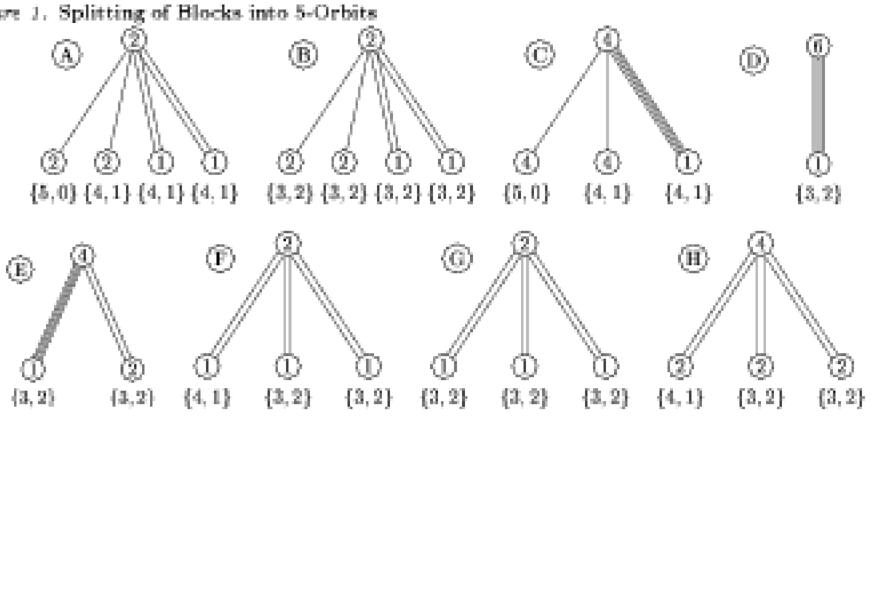

Table 2. The Inclusion Relation of 5-Orbits and Blocks (of Size 48 × 15)

orbit number	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
stabilizer order	2	2	2	2	4	2	2	2	2	4	2	6	2	4	2
additional point	25	30	26	30	34	24	28	35	36	21	19	31	32	31	33
split type	A	A	A	F	C	G	G	G	B	H	B	D	F	E	B
5-orbits															
$\{1,2,3,4,8\}_2$	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$\{1,2,3,4,5\}_2$	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0
$\{1,2,3,5,9\}_2$	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0
$\{1,2,3,6,26\}_1$	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0
$\{1,2,3,4,7\}_4$	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0
$\{1,2,3,22,23\}_1$	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0
$\{1,2,3,22,25\}_1$	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0
$\{1,2,3,22,31\}_1$	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0
$\{1,2,3,19,22\}_1$	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0
$\{1,2,3,4,19\}_2$	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0
$\{1,2,3,19,23\}_1$	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0
$\{1,2,3,24,25\}_1$	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0
$\{1,2,3,5,27\}_1$	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0
$\{1,2,3,19,27\}_1$	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0
$\{1,2,3,19,25\}_1$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1
$\{1,2,3,6,19\}_1$	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$\{1,2,3,4,26\}_2$	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$\{1,2,3,5,19\}_1$	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$\{1,2,3,6,22\}_2$	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0
$\{1,2,3,5,22\}_1$	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0
$\{1,2,3,4,23\}_1$	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0
$\{1,2,3,6,23\}_1$	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0
$\{1,2,3,5,24\}_1$	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0
$\{1,2,3,5,28\}_2$	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0
$\{1,2,3,23,33\}_1$	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0
$\{1,2,3,23,28\}_1$	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0
$\{1,2,3,6,25\}_1$	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0
$\{1,2,3,4,25\}_4$	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0
$\{1,2,3,22,24\}_1$	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0
$\{1,2,3,23,24\}_1$	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0
$\{1,2,3,22,26\}_1$	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0
$\{1,2,3,23,31\}_1$	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0
$\{1,2,3,22,32\}_1$	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0
$\{1,2,3,23,27\}_1$	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0
$\{1,2,3,22,29\}_2$	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0
$\{1,2,3,23,30\}_2$	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0
$\{1,2,3,19,26\}_1$	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0
$\{1,2,3,19,29\}_2$	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0
$\{1,2,3,19,20\}_2$	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0
$\{1,2,3,24,34\}_2$	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0
$\{1,2,3,19,24\}_1$	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0
$\{1,2,3,23,35\}_2$	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0
$\{1,2,3,23,25\}_1$	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0
$\{1,2,3,23,34\}_1$	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0
$\{1,2,3,24,33\}_2$	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0
$\{1,2,3,24,27\}_2$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1
$\{1,2,3,23,26\}_2$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1
$\{1,2,3,19,28\}_1$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1

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