Simple 8-(40,11,1440) Designs

Anton Betten, Reinhard Laue, Alfred Wassermann

Mathematical Department, University of Bayreuth, D-95440 Bayreuth, Germany

Abstract

In this short note simple 8-(40,11,1440) designs with automorphism group PSL(4,3) are presented. The designs are constructed with the method of Kramer and Mesner on a computer using the software package DISCRETA [2].

Key words: t-designs, group actions, constructive combinatorics

A simple t-(v, k, λ) design $\mathcal{D} = (\mathcal{V}, \mathcal{B})$ is a set $\mathcal{B}$ of k-subsets of a v-set $\mathcal{V}$ such that each t-subset of $\mathcal{V}$ is contained exactly λ times in $\mathcal{B}$. A recent overview on existence results of t-designs can be found in [6]. In this paper, we show the existence of 8-(40,11,1440) designs with automorphism group $A =$ PSL(4,3). The construction applies the method of Kramer and Mesner [5] using the software package DISCRETA [2]. There exist more than 100000 designs with this set of parameters and this group of automorphisms. We were not able to construct all designs, yet. By List [8] or Liebeck, Praeger, Saxl [7] the only overgroups of PSL(4,3) in S_{40} are A_{40} and PGL(4,3). Both are no automorphism groups of any 8-(40,11,1440) design. So, the full automorphism group of these designs is PSL(4,3) and the isomorphism types are the orbits of the normalizer PGL(4,3) on the set of these designs. Since each such orbit has size 2, there exist more than 50000 isomorphism types.

In a first step we constructed the Kramer-Mesner matrix $M_{t,k}^{A}$ for 8-(40,11,λ) designs by prescribing the group PSL(4,3) as group of automorphisms. For the definition of Kramer-Mesner matrices see [5].

The group $\mathrm{PSL}(4, 3)$ of order 6065280 is generated by the following permutations, which can be found in the list presented in the CRC Handbook of Combinatorial Designs [3, p. 603]:

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(1 3 5 7 9 11 13 15 17 19 21 23 25 27 29 31 33 35 37 39)
(2 4 6 8 10 12 14 16 18 20 22 24 26 28 30 32 34 36 38 40),
(1 29 26 4 13) (2 28 38 30 22)(3 8 39 16 32)(5 7 27 14 15)
(6 25 23 37 9)(10 17 11 40 35)(12 34 18 24 31)(19 36 21 33 20)

This group has the following number of orbits on s-subsets of the set $\mathcal{V} = \{1, 2, \ldots, 40\}$:

s	0	1	2	3	4	5	6	7	8	9	10	11
# s-orbits	1	1	1	2	4	6	12	24	53	111	263	569

Thus, the Kramer-Mesner matrix $M^A_{t,k}$ for 8-$(40, 11, \lambda)$ designs has 53 rows and 569 columns. Now, the theorem of Kramer-Mesner [5] tells that each 8-(40,11,1440) design with automorphism group PSL(4,3) corresponds to a solution of the diophantine linear system

$$M^A_{t,k} \cdot x = (\lambda, \lambda, \ldots, \lambda)^\top, \quad \text{where } x \in \{0, 1\}^{569}. \tag{1}$$

In a second step this system (1) was solved by an explicit enumeration algorithm based on lattice basis reduction as described in [11]. This algorithm quickly found more than 100000 solutions. In this note, we list the orbit representatives of the first solution of (1), given by the algorithm. For each orbit we give the lexicographically minimal representative and a subscript indicates the stabilizer order of this set. Each representative consists of an intervall $1, \ldots, i$ and some further points. Thus, only i and those further points are listed. As an example, the full base block of the first orbit $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 17\}_2$ is represented as $\{., 10, 17\}_2$. Its stabilizer order is equal to 2, which means that the length of this orbit is $6065280/2 = 3032640$.

This design consists of 178 orbits and the number of blocks equals $b =$ 671 168160.

Finally, we give some information on intersection numbers for this specific parameter set. Intersection numbers of designs can be used as a tool for classification. If one finds different intersection numbers, it can be deduced that the corresponding designs are non-isomorphic. A more extensive treatment of intersection numbers in that context can be found in [1].

The base blocks of the selected 8-(40,11,1440) design.

$\{., 10, 17\}_2$
$\{., 10, 19\}_1$
$\{., 10, 20\}_1$
$\{., 10, 35\}_2$
$\{., 9, 11, 16\}_1$
$\{., 9, 11, 19\}_1$
$\{., 9, 11, 20\}_1$
$\{., 9, 11, 23\}_2$
$\{., 9, 11, 24\}_2$
$\{., 9, 11, 30\}_1$
$\{., 9, 11, 33\}_1$
$\{., 9, 11, 34\}_2$
$\{., 9, 11, 37\}_1$
$\{., 9, 12, 16\}_2$
$\{., 9, 14, 17\}_1$
$\{., 9, 14, 18\}_1$
$\{., 9, 14, 26\}_1$
$\{., 9, 14, 34\}_1$
$\{., 9, 14, 36\}_1$
$\{., 9, 15, 19\}_1$
$\{., 9, 15, 24\}_1$
$\{., 9, 15, 33\}_2$
$\{., 9, 15, 35\}_1$
$\{., 9, 15, 38\}_2$
$\{., 9, 15, 39\}_1$
$\{., 9, 16, 17\}_2$
$\{., 9, 16, 18\}_2$
$\{., 9, 17, 23\}_2$
$\{., 9, 17, 35\}_2$
$\{., 9, 18, 27\}_2$
$\{., 9, 19, 25\}_4$
$\{., 9, 19, 34\}_1$
$\{., 9, 19, 40\}_1$
$\{., 9, 20, 21\}_2$
$\{., 9, 20, 34\}_2$
$\{., 9, 20, 36\}_2$
$\{., 9, 24, 25\}_4$
$\{., 9, 26, 40\}_4$
$\{., 9, 26, 33\}_1$
$\{., 9, 27, 34\}_2$
$\{., 9, 27, 37\}_1$
$\{., 9, 29, 35\}_2$
$\{., 9, 29, 37\}_2$
$\{., 9, 37, 38\}_8$

$\{., 8, 10, 11, 12\}_2$
$\{., 8, 10, 12, 15\}_1$
$\{., 8, 10, 12, 16\}_1$
$\{., 8, 10, 12, 19\}_1$
$\{., 8, 10, 12, 26\}_2$
$\{., 8, 10, 12, 34\}_2$
$\{., 8, 10, 14, 22\}_1$
$\{., 8, 10, 14, 34\}_1$
$\{., 8, 10, 15, 25\}_2$
$\{., 8, 10, 16, 26\}_1$
$\{., 8, 10, 17, 19\}_2$
$\{., 8, 10, 17, 26\}_2$
$\{., 8, 10, 17, 39\}_2$
$\{., 8, 11, 12, 19\}_6$
$\{., 8, 11, 15, 16\}_2$
$\{., 8, 11, 15, 39\}_2$
$\{., 8, 11, 20, 21\}_6$
$\{., 8, 14, 15, 27\}_1$
$\{., 8, 14, 15, 33\}_1$
$\{., 8, 14, 16, 20\}_1$
$\{., 8, 14, 16, 26\}_1$
$\{., 8, 14, 16, 27\}_1$
$\{., 8, 14, 17, 27\}_3$
$\{., 8, 14, 19, 20\}_2$
$\{., 8, 14, 19, 26\}_2$
$\{., 8, 14, 19, 28\}_2$
$\{., 8, 14, 20, 24\}_2$
$\{., 8, 14, 20, 26\}_2$
$\{., 8, 14, 20, 39\}_1$
$\{., 8, 14, 24, 35\}_8$
$\{., 8, 14, 26, 28\}_2$
$\{., 8, 15, 16, 27\}_2$
$\{., 8, 15, 16, 33\}_1$
$\{., 8, 15, 17, 20\}_2$
$\{., 8, 15, 17, 33\}_6$
$\{., 8, 15, 17, 37\}_6$
$\{., 8, 15, 20, 37\}_6$
$\{., 7, 9, 11, 13, 17\}_1$
$\{., 7, 9, 11, 13, 19\}_2$
$\{., 7, 9, 11, 13, 27\}_2$
$\{., 7, 9, 11, 14, 19\}_2$
$\{., 7, 9, 11, 15, 19\}_2$
$\{., 7, 9, 11, 16, 28\}_{12}$
$\{., 7, 9, 11, 17, 19\}_2$
$\{., 7, 9, 11, 19, 22\}_8$

$\{., 7, 9, 13, 15, 19\}_1$
$\{., 7, 9, 13, 15, 22\}_1$
$\{., 7, 9, 13, 15, 25\}_1$
$\{., 7, 9, 13, 15, 26\}_2$
$\{., 7, 9, 13, 15, 27\}_1$
$\{., 7, 9, 13, 15, 39\}_1$
$\{., 7, 9, 13, 17, 27\}_2$
$\{., 7, 9, 13, 17, 33\}_1$
$\{., 7, 9, 13, 17, 34\}_2$
$\{., 7, 9, 13, 17, 35\}_1$
$\{., 7, 9, 13, 17, 38\}_1$
$\{., 7, 9, 13, 19, 27\}_1$
$\{., 7, 9, 13, 19, 33\}_1$
$\{., 7, 9, 13, 19, 38\}_6$
$\{., 7, 9, 13, 20, 32\}_6$
$\{., 7, 9, 13, 20, 38\}_2$
$\{., 7, 9, 13, 22, 27\}_1$
$\{., 7, 9, 14, 15, 35\}_1$
$\{., 7, 9, 14, 35, 38\}_{12}$
$\{., 7, 9, 15, 16, 17\}_1$
$\{., 7, 9, 15, 16, 19\}_2$
$\{., 7, 9, 15, 19, 26\}_1$
$\{., 7, 9, 15, 25, 28\}_2$
$\{., 7, 9, 16, 19, 20\}_{12}$
$\{., 7, 9, 17, 19, 22\}_1$
$\{., 6, 8, 9, 10, 11, 16\}_1$
$\{., 6, 8, 9, 10, 11, 18\}_1$
$\{., 6, 8, 9, 10, 11, 19\}_1$
$\{., 6, 8, 9, 10, 11, 25\}_1$
$\{., 6, 8, 9, 10, 11, 31\}_1$
$\{., 6, 8, 9, 10, 12, 15\}_3$
$\{., 6, 8, 9, 10, 12, 18\}_1$
$\{., 6, 8, 9, 10, 12, 25\}_1$
$\{., 6, 8, 9, 10, 12, 32\}_1$
$\{., 6, 8, 9, 10, 12, 34\}_1$
$\{., 6, 8, 9, 10, 12, 39\}_1$
$\{., 6, 8, 9, 10, 14, 19\}_2$
$\{., 6, 8, 9, 10, 14, 23\}_1$
$\{., 6, 8, 9, 10, 14, 30\}_1$
$\{., 6, 8, 9, 10, 14, 39\}_1$
$\{., 6, 8, 9, 10, 15, 33\}_1$
$\{., 6, 8, 9, 10, 16, 28\}_2$
$\{., 6, 8, 9, 10, 17, 18\}_2$
$\{., 6, 8, 9, 10, 17, 29\}_1$
$\{., 6, 8, 9, 10, 20, 39\}_2$

$\{., 6, 8, 9, 10, 23, 31\}_1$
$\{., 6, 8, 9, 10, 25, 31\}_2$
$\{., 6, 8, 9, 10, 29, 39\}_3$
$\{., 6, 8, 9, 10, 30, 31\}_1$
$\{., 6, 8, 9, 10, 31, 32\}_1$
$\{., 6, 8, 9, 11, 15, 25\}_1$
$\{., 6, 8, 9, 11, 15, 26\}_2$
$\{., 6, 8, 9, 11, 15, 32\}_1$
$\{., 6, 8, 9, 11, 17, 29\}_3$
$\{., 6, 8, 9, 11, 18, 23\}_2$
$\{., 6, 8, 9, 11, 18, 34\}_1$
$\{., 6, 8, 9, 11, 19, 20\}_1$
$\{., 6, 8, 9, 11, 19, 34\}_2$
$\{., 6, 8, 9, 11, 21, 32\}_1$
$\{., 6, 8, 9, 11, 25, 31\}_{12}$
$\{., 6, 8, 9, 11, 30, 38\}_1$
$\{., 6, 8, 9, 11, 31, 32\}_6$
$\{., 6, 8, 9, 11, 31, 38\}_6$
$\{., 6, 8, 9, 12, 17, 20\}_2$
$\{., 6, 8, 9, 12, 21, 38\}_2$
$\{., 6, 8, 9, 12, 32, 37\}_8$
$\{., 6, 8, 9, 14, 23, 32\}_2$
$\{., 6, 8, 9, 14, 25, 26\}_8$
$\{., 6, 8, 9, 14, 30, 37\}_6$
$\{., 6, 8, 9, 14, 32, 33\}_6$
$\{., 6, 8, 9, 16, 17, 20\}_2$
$\{., 6, 8, 9, 16, 17, 37\}_6$
$\{., 6, 8, 9, 17, 18, 29\}_2$
$\{., 6, 8, 9, 17, 29, 34\}_8$
$\{., 6, 8, 9, 18, 29, 34\}_8$
$\{., 6, 8, 9, 19, 21, 25\}_8$
$\{., 6, 8, 10, 14, 28, 32\}_{24}$
$\{., 6, 8, 11, 12, 19, 21\}_4$
$\{., 6, 8, 11, 12, 19, 24\}_6$
$\{., 6, 10, 11, 14, 15, 26\}_6$
$\{., 6, 10, 11, 14, 15, 30\}_{12}$
$\{., 6, 10, 11, 18, 19, 20\}_{36}$
$\{., 6, 10, 11, 18, 19, 23\}_{12}$
$\{., 6, 10, 11, 18, 20, 23\}_{36}$
$\{., 6, 8, 17, 24, 27, 29\}_{24}$
$\{., 6, 8, 19, 21, 28, 29\}_{12}$
$\{., 6, 10, 14, 15, 16, 26\}_4$
$\{., 5, 8, 9, 16, 18, 34, 38\}_{108}$
$\{., 4, 6, 9, 10, 17, 18, 22, 35\}_{108}$

For subsets I and J of $\mathcal{V}$ with $|I| = i$, $|J| = j$ and $0 \leq i + j \leq t$ define

$$\lambda_{i,j} := |\{B \in \mathcal{B} \mid I \subseteq B \wedge J \cap B = \emptyset\}|.$$

Then the intersection triangle $\lambda_{i,j}$ with $0 \leq i+j \leq t$ for 8-(40,11,1440) designs is the following:

671168160	486596916	349351632	248223528	174427344	121130100	83060640	56188080	37458720
184571244	137245284	101128104	73796184	53297244	38069460	26872560	18729360	
47325960	36117180	27331920	20498940	15227784	11196900	8143200		
11208780	8785260	6832980	5271156	4030884	3053700			
2423520	1952280	1561824	1240272	977184				
471240	390456	321552	263088					
80784	68904	58464						
11880	10440							
1440								

For an arbitrary fixed m-subset M of $\mathcal{V}$ the j-th *intersection number* of M with the design $\mathcal{D}$ is defined for $0 \leq j \leq m$ as

$$\alpha_j(M) := \Big|\{B \in \mathcal{B} \mid |M \cap B| = j\}\Big|.$$

Mendelsohn [9] gave the equations

$$\sum_{j=i}^{k} \binom{j}{i} \alpha_j(M) = \binom{m}{i} \lambda_{i,0} \quad \text{for all } i = 0, 1, \dots, t. \tag{2}$$

Köhler [4] expressed $\alpha_0(M)$, $\alpha_1(M)$, … , $\alpha_t(M)$ in terms of $\alpha_{t+1}(M)$, … , $\alpha_m(M)$. Here, for $M \in \mathcal{B}$, i.e. $m = k$, Köhler's equations read as:

$$\begin{aligned}
\alpha_0(M) &= 10\,051\,704 - 1\,\alpha_9(M) - 9\,\alpha_{10}(M) - 45\,\alpha_{11}(M)\\
\alpha_1(M) &= 63\,900\,936 + 9\,\alpha_9(M) + 80\,\alpha_{10}(M) + 396\,\alpha_{11}(M)\\
\alpha_2(M) &= 160\,180\,020 - 36\,\alpha_9(M) - 315\,\alpha_{10}(M) - 1540\,\alpha_{11}(M)\\
\alpha_3(M) &= 204\,995\,340 + 84\,\alpha_9(M) + 720\,\alpha_{10}(M) + 3465\,\alpha_{11}(M)\\
\alpha_4(M) &= 150\,448\,320 - 126\,\alpha_9(M) - 1050\,\alpha_{10}(M) - 4950\,\alpha_{11}(M)\\
\alpha_5(M) &= 62\,802\,432 + 126\,\alpha_9(M) + 1008\,\alpha_{10}(M) + 4620\,\alpha_{11}(M)\\
\alpha_6(M) &= 16\,532\,208 - 84\,\alpha_9(M) - 630\,\alpha_{10}(M) - 2772\,\alpha_{11}(M)\\
\alpha_7(M) &= 2\,019\,600 + 36\,\alpha_9(M) + 240\,\alpha_{10}(M) + 990\,\alpha_{11}(M)\\
\alpha_8(M) &= 237600 - 9\,\alpha_9(M) - 45\,\alpha_{10}(M) - 165\,\alpha_{11}(M)
\end{aligned} \tag{3}$$

Note that $\alpha_{11}(M) = 1$ if and only if M is a block of the design. (Otherwise, $\alpha_{11}(M) = 0$).

The *global intersection numbers* $\alpha_j^{(s)}(\mathcal{D})$ *of order* s of the design $\mathcal{D}$ are:

$$\alpha_j^{(s)}(\mathcal{D}) := \left|\left\{\{B_{i_1},\dots,B_{i_s}\} \in \binom{\mathcal{B}}{s} \;\Big|\; |B_{i_1}\cap\dots\cap B_{i_s}| = j\right\}\right|.$$

In [1] it is shown that these global intersection numbers satisfy the following system of equations

$$\sum_{j=i}^{k}\binom{j}{i}\alpha_j^{(s)}(\mathcal{D}) = \binom{v}{i}\binom{\lambda_{i,0}}{s} \quad \text{for all } i = 0,1,\dots,t$$

which is derived from the generalized MENDELSOHN system [9,10]. The main source of information on higher order intersection numbers is [10]. For $s = 1$ these equations are exactly the equations (2). For higher values of s only the right hand side of the systems differs. As in (3), $\alpha_0^{(s)}(\mathcal{D}), \dots, \alpha_t^{(s)}(\mathcal{D})$ can be expressed according to [4] and [10] in terms of $\alpha_{t+1}^{(s)}(\mathcal{D}), \dots, \alpha_{k-1}^{(s)}(\mathcal{D})$ (note that $\alpha_k^{(s)}(\mathcal{D}) = 0$ for $s > 1$). Here, we give the equations for $s = 2$:

$$\begin{aligned}
\alpha_0^{(2)}(\mathcal{D}) &= 3373\,176737\,988720 - 1\,\alpha_9^{(2)}(\mathcal{D}) - 9\,\alpha_{10}^{(2)}(\mathcal{D})\\
\alpha_1^{(2)}(\mathcal{D}) &= 21444\,269709\,994560 + 9\,\alpha_9^{(2)}(\mathcal{D}) + 80\,\alpha_{10}^{(2)}(\mathcal{D})\\
\alpha_2^{(2)}(\mathcal{D}) &= 53753\,347846\,598400 - 36\,\alpha_9^{(2)}(\mathcal{D}) - 315\,\alpha_{10}^{(2)}(\mathcal{D})\\
\alpha_3^{(2)}(\mathcal{D}) &= 68794\,335377\,024400 + 84\,\alpha_9^{(2)}(\mathcal{D}) + 720\,\alpha_{10}^{(2)}(\mathcal{D})\\
\alpha_4^{(2)}(\mathcal{D}) &= 50486\,399913\,549600 - 126\,\alpha_9^{(2)}(\mathcal{D}) - 1050\,\alpha_{10}^{(2)}(\mathcal{D})\\
\alpha_5^{(2)}(\mathcal{D}) &= 21077\,046762\,932160 + 126\,\alpha_9^{(2)}(\mathcal{D}) + 1008\,\alpha_{10}^{(2)}(\mathcal{D})\\
\alpha_6^{(2)}(\mathcal{D}) &= 5547\,015572\,978880 - 84\,\alpha_9^{(2)}(\mathcal{D}) - 630\,\alpha_{10}^{(2)}(\mathcal{D})\\
\alpha_7^{(2)}(\mathcal{D}) &= 678\,077836\,207200 + 36\,\alpha_9^{(2)}(\mathcal{D}) + 240\,\alpha_{10}^{(2)}(\mathcal{D})\\
\alpha_8^{(2)}(\mathcal{D}) &= 79\,679406\,034800 - 9\,\alpha_9^{(2)}(\mathcal{D}) - 45\,\alpha_{10}^{(2)}(\mathcal{D})
\end{aligned}$$

For the above design, DISCRETA computed the following values:

$$\alpha_9^{(2)}(\mathcal{D}) = 2\,169140\,968800, \quad \alpha_{10}^{(2)}(\mathcal{D}) = 30924\,644400\,.$$

These global intersection numbers $\left(\alpha_9^{(2)}(\mathcal{D}), \alpha_{10}^{(2)}(\mathcal{D})\right)$ can be used as a fingerprint of a design. If for two designs $\mathcal{D}_1, \mathcal{D}_2$

$$\left(\alpha_9^{(2)}(\mathcal{D}_1), \alpha_{10}^{(2)}(\mathcal{D}_1)\right) \neq \left(\alpha_9^{(2)}(\mathcal{D}_2), \alpha_{10}^{(2)}(\mathcal{D}_2)\right),$$

then the designs are non-isomorphic. Among the first 400 solutions computed by DISCRETA there are 389 different values of $\left(\alpha_9^{(2)}(\mathcal{D}), \alpha_{10}^{(2)}(\mathcal{D})\right)$.

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