The Proper Linear Spaces on 17 Points

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Abstract

The proper linear spaces on 17 points are classified. The computation is based on the parameters of the geometries and makes extensive use of tactical decompositions. A specific one, the tactical decomposition by ordering (TDO) which has been invented by D. Betten and M. Braun in [5] is presented in full detail. The TDO may be seen as the final step of parameters of the geometries. In the current article, the authors show how the TDO can be used in order to construct geometries. This new method starts by calculating all possible TDO-schemes which the requested geometries may have. In a second step, all geometries for a fixed TDO-scheme are constructed. This two-step approach is a versatile tool which may be applied to other construction problems, too. The current work may be seen as an extension of [3] where all linear spaces on 12 points were classified.

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1 Introduction

A *linear space* is an incidence structure $S = (\mathcal{V}, \mathcal{B})$ with a set of points $\mathcal{V}$ and a set of subsetes of $\mathcal{V}$ called blocks or lines $\mathcal{B}$ such that the following two conditions are satisfied: Each line contains at least 2 points and each pair of points lies on a unique line. In this article, we require the set $\mathcal{V}$ to be finite and denote the number of points by v and the number of lines (or blocks) by b. A linear space is called *proper* if each line contains at least three points and no line has length v (the last condition just excludes the linear space consisting of a single line of length v).

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Whereas the number of linear spaces grows rapidly with v only few of them are proper. Ch. Pietsch showed in [16] that there are 232,929 linear spaces on 11 points. The current authors determined the 28,872,973 linear spaces on 12 points in [3]. All proper linear spaces on up to 16 points are known: A. Brouwer [6] computed them for $v \leq 15$ and G. Heathcote [14] did the case on 16 points. The current note handles the case $v = 17$.

Historically, linear spaces were studied among others by J. Doyen who enumerated all linear spaces on at most 9 points (see [8]). The linear spaces on 10 points were enumerated independently by D. Glynn [10] and the second author in 1990. The book by L. Batten and A. Beutelspacher [1] contains a lot of drawings of linear spaces on small point sets.

2 Some More Notions

A linear space is called *regular* if for each $j = 2, \ldots, v$ the number of lines of length j through a point p depends only on j. We denote this number by $[p]_j$ and call it the *j-degree* ($[p]$ is the usual degree, i. e., the number of lines through p). In [2], the current authors determine the regular linear spaces on up to 16 points with only few (but probably large) exceptional cases. In the meantime, one open case could be settled, compare the work by A. Betten, G. Brinkmann and T. Pisanski [4]. We denote the parameters of a regular linear space by $(v|\,[p]_2,\,[p]_3,\,\ldots\,,\,[p]_v)$ as introduced in [2]. We will also call this the *type* of the regular linear space. The choice of p is irrelevant according to the definition. Clearly, a regular linear space is proper if $[p]_2 = [p]_v = 0$. So, the proper regular linear spaces can be obtained by collecting complete subcases of the list of regular linear spaces. The number of regular linear spaces of type $(v|\,[p]_2,\,[p]_3,\,\ldots\,,\,[p]_v)$ is denoted by $\#(v|\,[p]_2,\,[p]_3,\,\ldots\,,\,[p]_v)$.

A linear space is an $(r,1)$*-design* if the degree of all points equals a constant r. H. Gropp has studied $(r,1)$-designs with at most 12 points in [13]. From

$$[p] \;=\; \sum_{j=2}^{v} [p]_j$$

we get that a regular linear space is also an $(r,1)$-design (the terms on the right are independent of the choice of p so is the term on the left).

Fundamental is the notion of a *configuration*. A configuration of type $v_r b_k$ (or: (v_r, b_k)) is an incidence geometry on v points with b blocks of size k such that each point has degree r and each pair of points is contained in at most one block. A configuration with $v = b$ (and hence also $r = k$ because of the well-known equation $vr = kb$ for configurations) is called *symmetric*. In this case, the type is simply indicated by v_r. The notions of regular linear spaces and configurations meet sometimes. For example, a configuration 16_3 is also a regular linear space with parameters $(16|72, 16)$. This is due to the fact that a configuration can be embedded uniquely into a linear space by adding lines of length two: if a pair of points is not yet joined in the configuration, join it by a 2-line. The totality of all these additional 2-lines forms a graph, the *configuration graph*. This graph is regular and its degree is called the *deficiency* of the configuration (compare Gropp [12]).

Table 1 shows the numbers of linear spaces known so far. The number 3,004,881 of spaces of type $(16|72, 16)$ is taken from [4]. The sequence of numbers of non-isomorphic linear spaces and proper linear spaces is contained in the article of H.-D. Gronau, R. Mullin and Ch. Pietsch in the Handbook of Combinatorial Designs [11]. The number of proper linear spaces on 17 points is new

Table 1
Numbers of Linear Spaces

v	total	proper	regular		proper regular
2	1	0	1		0
3	2	0	2		0
4	3	0	2		0
5	5	0	2		0
6	10	0	4		0
7	24	1	3		1
8	69	0	4		0
9	384	1	9		1
10	5,250	1	14		0
11	232,929	1	33		0
12	28,872,973	3	839		3
13		7	2,041		3
14		1	22,192		0
15		119	245,773	$+ \# (15\|15, 30)$	84
				$+ \# (15\|30, 25)$	
				$+ \# (15\|45, 20)$	
16		398	3,306,477	$+ \# (16\|24, 32)$	25
17		161,925			0

and will be presented in more detail in Section 7. The following cases of regular linear spaces on 15 and 16 points, which are in fact configurations are still open: $(15|15, 30) \hat{=} 15_6 30_3$, $(15|30, 25) \hat{=} 15_5 25_3$, $(15|45, 20) \hat{=} 15_4 20_3$ and $(16|24, 32) \hat{=} 16_6 32_3$.

3 Tactical Decompositions of Finite Incidence Structures

The kind of geometries we are interested in is a very general class of species. Dembowski in his influential book [7, Section 1.1] defined parameters of incidence geometries according to regularity conditions $(R.m)$ and $(\overline{R}.n)$. These conditions ensure that the number of points which are common to m blocks (or the number of blocks which all contain n fixed points) is nonzero and does not depend on the choice of the m-subset of blocks (or n-subset of points). Thus, $(R.0)$ and $(\overline{R}.0)$ are equivalent to $\mathcal{V} \neq \emptyset$ and $\mathcal{B} \neq \emptyset$. $(R.1)$ means that all blocks are non-empty and have the same size. $(\overline{R}.1)$ means that all points have the same nonzero degree. Dembowski introduces integers s and t and calls (s, t) the *type* of the incidence geometry. The integers s and t shall be maximal with the property that $(R.1), \ldots, (R.s)$ and $(\overline{R}.1), \ldots, (\overline{R}.t)$ are satisfied but $(R.s+1)$ and $(\overline{R}.t+1)$ are violated. In this notation, a t-design which is not a $(t+1)$-design is exactly an incidence geometry of type $(1, t)$. He shows that up to duality t-designs with large t are of particular interest.

By definition, a linear space satisfies $(R.0)$ and $(\overline{R}.0)$. As each pair of points is contained in exactly one block, $(\overline{R}.2)$ is fulfilled, too. But without $(\overline{R}.1)$ we only have the type $(0, 0)$ for linear spaces in general. We may get $(0, 2)$ if every point is contained in the same number of blocks which is true for $(r, 1)$-designs. If in addition all blocks have the same size, we get the type $(1, 2)$ and this leads to 2-design. Note that even the type $(2, 2)$ is possible: In this case, the linear space is a symmetric design with $\lambda = 1$, i. e., a projective plane.

Let us begin with some introductory remarks on tactical decompositions and applications on proper linear spaces.

Let $\mathcal{V} = \{p_1, \ldots, p_v\}$ be the set of points and $\mathcal{B} = \{B_1, \ldots, B_n\}$ be the set of blocks of a finite incidence structure $S = (\mathcal{V}, \mathcal{B})$. For a finite set M we call any decomposition of M into disjoint subsets $P_1, \ldots, P_g$ such that $M = \cup_{i=1}^{g} P_i$ a *set partition* of M and write $P \vdash M$. A *decomposition* of the incidence structure S is a set partition $P = (P_1, \ldots, P_g) \vdash \mathcal{V}$ together with another set partition $Q = (Q_1, \ldots, Q_h) \vdash \mathcal{B}$. Moreover, the decomposition is said to be *point-tactical* if for any $i \leq g$ and any $j \leq h$ the number

$$\alpha_{i,j} \ = \ |\{ B \in Q_j \mid p \in B \}| \tag{1}$$

is independent of the choice of the point $p \in P_i$ (thus, the number depends only on i and j and — clearly — the decompositions P and Q). The decomposition is *block-tactical* if for any $i \leq g$ and any $j \leq h$ the number

$$\beta_{i,j} = |\{ p \in P_i \mid p \in B \}| \tag{2}$$

is independent of the choice of $B \in Q_j$. A decomposition which is both, point- and block-tactical is simply called *tactical*. In this case we get two integral matrices $A = (\alpha_{i,j})$ and $B = (\beta_{i,j})$ describing the decomposition. Together with the sizes of the point- and block-classes $(|P_1|, \ldots, |P_g|)$, $(|Q_1|, \ldots, |Q_h|)$, we call this the *scheme* of the decomposition. For a tactical decomposition, the equation

$$|P_i| \cdot \alpha_{i,j} = \beta_{i,j} \cdot |Q_j| \tag{3}$$

holds for all $i = 1, \ldots, g$, $j = 1, \ldots, h$.

A *trivial decomposition* is obtained by the discrete partition of points and blocks, where each point and block forms a class of its own. In this case, the numbers above coincide and we get $\alpha_{i,j} = \beta_{i,j} = m_{i,j}$ where $m_{i,j}$ is the (i,j)-th entry of the 0/1-incidence matrix, which is 1 if and only if $p_i \in B_j$ holds. We call this the *discrete decomposition*.

Another trivial decomposition exists for any incidence geometry $S = (\mathcal{V}, \mathcal{B})$. We put $g = h = 1$ with $P_1 = \mathcal{V}$ and $Q_1 = \mathcal{B}$ and obtain the decomposition $P = (P_1)$ of points and $Q = (Q_1)$ of lines. This decomposition is tactical if and only if the column and row sums of the incidence matrix are constants, usually denoted by k and r, respectively. In this case, the decomposition schemes are

$$\begin{array}{r|l} \alpha_{i,j} & b \\ \hline v & r \end{array} \quad \text{and} \quad \begin{array}{r|l} \beta_{i,j} & b \\ \hline v & k \end{array}, \tag{4}$$

where we indicate in the top left position which kind of numbers is contained in the scheme. We call this last decomposition the *all-in-one decomposition*.

In the sequel, we will often show incidence matrices for proper linear spaces. As such a matrix is not unique we will always show *canonical incidence matrices* for the geometries. This means that a specific permutation of points and blocks is applied which takes the incidence matrix into its column-lexicographic maximal form.

Example 1 The smallest proper linear space is the configuration 7_3, the projective plane of order 2 (which has an automorphism group of order 168). In Figure 1, we have the famous picture of this plane together with the canonical incidence matrix and the all-in-one decomposition. ◇

Example 2 As there is no proper linear space on 8 points the next example

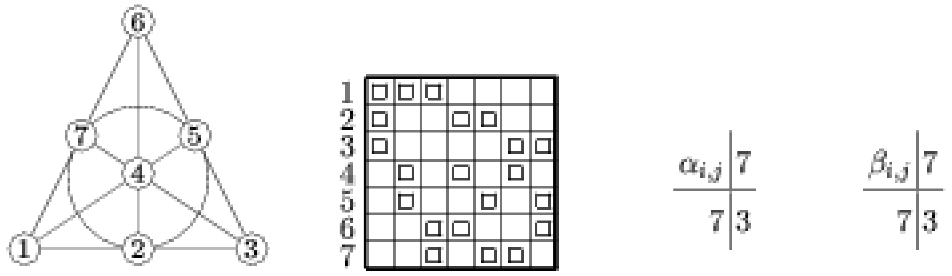


Fig. 1. The Fano-Plane

is the unique one on 9 points. It is better known as the affine plane of order 3 and has an automorphism group of order 432. This proper linear space is also a configuration $9_4 12_3$, and we have the all-in-one decomposition in this case as well. In Figure 2, we show a drawing together with the incidence matrix and the decomposition schemes. Clearly, $\beta_{1,1} = |P_1| \cdot \alpha_{1,1}/|Q_1| = 9 \cdot 4/12 = 3$ according to Equation (3). ◇

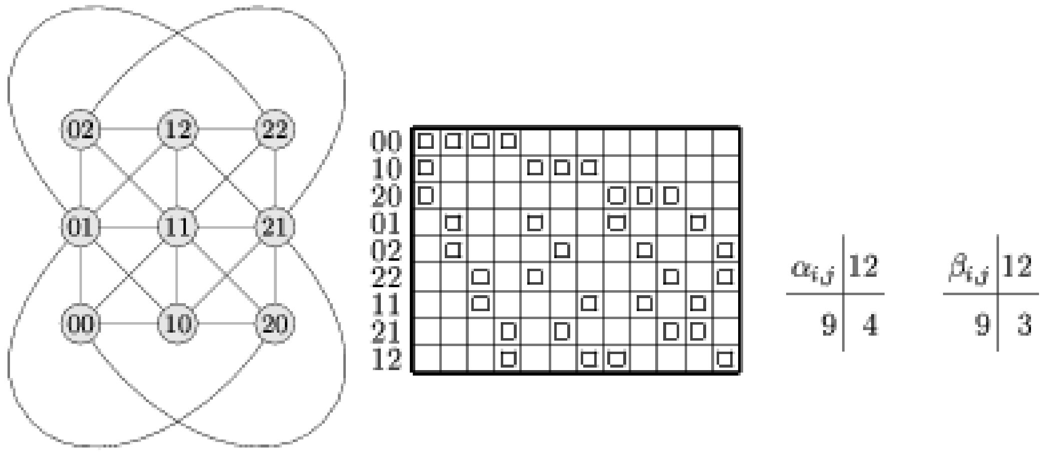


Fig. 2. The Affine Plane of Order 3

In the following, we will make use of tactical decompositions of all kinds, in particular those which are point-tactical, but not block-tactical (or the other way round). We also need the well-known ordering of set partitions. Therefore, let $P, P' \vdash M$ be two partitions of M as a set with $P = (P_1, P_2, \ldots, P_g)$ and $P' = (P'_1, P'_2, \ldots, P'_h)$ and put

$$P \prec P' \;:\Leftrightarrow\; \forall i = 1, \ldots, g \;\; \exists j \leq h : \; P_i \subseteq P'_j. \tag{5}$$

We call P a *refinement* of P' as the classes of P' are unions of complete classes of P. We also say that P' is *coarser* than P. It can be shown that this ordering gives a *lattice of set partitions*. For any two set partitions $P, P' \vdash M$, the infimum $P \wedge P'$ is formed by the non-empty intersections of the classes of P with the classes of P' and the supremum $P \vee P'$ is the finest set partition of M whose classes can all be written as unions of whole classes of P and of whole classes of P'.

4 The Tactical Decomposition by Ordering

In general, proper linear spaces are incidence geometries without regularity apart from the trivial ones $(R.0)$ and $(\overline{R}.0)$ together with $(\overline{R}.2)$. In order to study this quite general class of objects we employ the concept of tactical decompositions introduced in the last section. A refined version, namely the canonical tactical decomposition which is obtained by ordering (the so-called TDO) has been introduced by D. Betten and M. Braun in [5]. This algorithmic tool will be presented in the current section.

The TDO-method leads to a decomposition of the incidence geometry and thus comes together with integral matrices describing the decomposition, the so-called *TDO-scheme*. The TDO can be computed for every finite incidence geometry. Its most useful property is that it is invariant under isomorphism of incidence geometries and may thus be used for classification purposes. The concept of TDO-decomposition is based on the parameters of the geometry which are easily obtained by counting. In order to get a tactical decomposition one goes on with the parameters as far as possible, i. e., till no two points or blocks can be distinguished within the decomposition. This means that the resulting decomposition is tactical. The TDO-scheme may thus be seen as the final step of parameters for the geometry (we will discuss parameters of geometries in more detail in Section 5).

In order to compute a transversal of the isomorphism classes of a given set of geometries, one may apply the TDO in two different ways. First, it is a handy tool to verify quickly that two given geometries are non-isomorphic. On the other hand, one may also speed up the search for isomorphisms or automorphisms of geometries. The TDO partitions points and blocks of the geometry and the classes of such a partition are preserved by the group of automorphisms.

The TDO is obtained by application of the TDO-algorithm. In order to describe this procedure briefly, it is useful to note that an arbitrary decomposition of points and blocks of an incidence matrix can be refined to become either point- or block-tactical. Here and in the following, we identify an incidence structure on a set of points $\mathcal{V} = \{p_1, \ldots, p_v\}$ with its incidence matrix $\mathbf{I}$. We write $S = (\mathcal{V}, \mathcal{B}, \mathbf{I})$ and identify rows and columns of $\mathbf{I}$ with points and blocks, respectively.

Lemma 1 *Let $\mathbf{I}$ be an incidence matrix of size $v \times b$ and $P \vdash \mathcal{V}$ and $Q \vdash \mathcal{B}$ be arbitrary decompositions of rows (points) and columns (blocks) respectively. Then there exists a decomposition $P' \prec P$ such that (P', Q) is point-tactical. Moreover, the coarsest P' with this property is uniquely determined. On the other hand, there also exists a refinement $Q' \prec Q$ such that (P, Q') is block-*

tactical. The coarsest Q' with this property is uniquely determined.

PROOF. Let P and Q have g and h classes respectively. Call them $P_1, P_2, \ldots, P_g$ and $Q_1, Q_2, \ldots, Q_h$. Consider separately each class of the decomposition P. For each point p_i, compute the type $(t_1(p_i), t_2(p_i), \ldots, t_h(p_i))$, where

$$t_j(p_i) = |\{B \in Q_j \mid p_i \in B\}|$$

for $j = 1, \ldots, h$ is the number of incidences of p_i with blocks of Q_j. Within the classes of the original decomposition P, we now reorder rows according to these type vectors. This is done by sorting the rows with respect to their type vectors in a lexicographic manner. Collecting rows of equal type and putting them together into classes we obtain a finer decomposition $P' \prec P$.

As the rows of each class of P' have equal type vectors, the decomposition (P', Q) is point-tactical. Clearly, it is the coarsest one with this property refining P. The second statement about the existence of a refinement $Q' \prec Q$ follows by duality.

A few remarks are in order: First, we would like to stress that the specific lexicographic ordering which is applied for rearranging rows is not too important (whether we sort lexicographically increasing or decreasing, for example). Obviously, the resulting decompositions differ in their ordering of classes but this does not bother us too much. What is really important is that we always get the same result from this algorithm when applying a fixed ordering. We will come back to this point later.

In addition, it should be remarked that when starting with a block-tactical decomposition (P, Q) the resulting point-tactical decomposition (P', Q) obtained from Lemma 1 may lack the property of being block-tactical (clearly, the dual statement also holds for block-refinements and tacticality on points). Thus, decomposing in one direction may destroy tacticality in the orthogonal direction.

Now, we can proceed to describe the TDO procedure. In order to get the TDO, we start with the trivial all-in-one decomposition of points and blocks consisting of a single class each. We are now going to refine the decompositions of points and blocks in an alternating manner. In each of these steps of refinement, we either obtain a point-tactical decomposition of points or a block-tactical decomposition of blocks. This refinement is based on Lemma 1. More precisely, we employ the following algorithm, which we call TDO-algorithm:

Put $P^{(1)} := (\mathcal{V})$ and $Q^{(1)} := (\mathcal{B})$ the all-in-one decomposition. If the decomposition $(P^{(1)}, Q^{(1)})$ is not point-tactical we refine $P^{(1)}$ and get a new decomposition $P^{(2)} \vdash \mathcal{V}$ with $P^{(2)} \prec P^{(1)}$. This is done in such a way that $(P^{(2)}, Q^{(1)})$ is point-tactical and $P^{(2)}$ is the coarsest decomposition refining $P^{(1)}$ with this property (such a refinement exists and is unique according to Lemma 1). If $(P^{(2)}, Q^{(1)})$ is also block-tactical, we put $Q^{(2)} := Q^{(1)}$. Otherwise, we determine $Q^{(2)} \prec Q^{(1)}$ such that $(P^{(2)}, Q^{(2)})$ is block-tactical and $Q^{(2)}$ is the coarsest refinement with this property. At this point, we have obtained another decomposition $(P^{(2)}, Q^{(2)})$ of S which is not necessarily point-tactical. We may now iterate these two steps of refinement forming new decompositions $P^{(i+1)} \prec P^{(i)}$ and $Q^{(i+1)} \prec Q^{(i)}$ for $i = 2, 3, \ldots$. Eventually, we arrive at a decomposition $(P^{(\ell)}, Q^{(\ell)})$ which is both, point- and block-tactical. We call this decomposition *TDO-decomposition*.

The fact that the TDO-algorithm really terminates after a finite number of steps follows from the assumption of S to be finite. In each step, at least one decomposition is refined strictly and thus, this process of refining the decompositions and reordering points and blocks will come to an end. In addition, the trivial discrete decomposition is tactical as we have pointed out previously.

The decomposition-matrices $A = (\alpha_{ij})$ and $B = (\beta_{ij})$ of the TDO-decomposition together with the vectors $(|P_1^{(\ell)}|, \ldots, |P_{g_\ell}^{(\ell)}|)$ and $(|Q_1^{(\ell)}|, \ldots, |Q_{h_\ell}^{(\ell)}|)$ form the *TDO-scheme*.

Next, we display the algorithm for computing the TDO in some abstract notation which is close to real programming languages.

Algorithm: computing TDO.
  input: A $v \times b$ incidence matrix $\mathbf{I}$ of an incidence structure $S = (\mathcal{V}, \mathcal{B}, \mathbf{I})$.
  output: an integer i and decompositions $P^{(i)} \vdash \mathcal{V}$ and $Q^{(i)} \vdash \mathcal{B}$.

TDO (v, b, I)
  int i;
  $i := 1$;
  $P^{(1)} := (\mathcal{V})$;
  $Q^{(1)} := (\mathcal{B})$;
  do
    if $(P^{(i)}, Q^{(i)})$ is point-tactical **then**
      $P^{(i+1)} := P^{(i)}$;
    else
      determine $P^{(i+1)} \prec P^{(i)}$ coarsest such that
      $(P^{(i+1)}, Q^{(i)})$ is point-tactical
      // such a refinement exists according to Lemma 1
    if $(P^{(i+1)}, Q^{(i)})$ is block-tactical **then**
      $Q^{(i+1)} := Q^{(i)}$;

```
        else
            determine Q^(i+1) ≺ Q^(i) coarsest such that
            (P^(i+1), Q^(i+1)) is block-tactical
            // such a refinement exists according to Lemma 1
        i := i + 1;
    until P^(i) = P^(i-1) and Q^(i) = Q^(i-1)
return i, P^(i), Q^(i); // this is the TDO decomposition
```

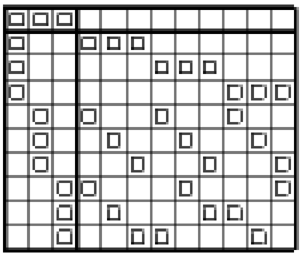
The most important property of the TDO-decomposition is the following: We call a decomposition *canonical*, if it is obtained by an algorithm yielding the same decomposition for every element of the same isomorphism class of incidence structures. The following lemma proves that the TDO-decomposition is canonical. This means that geometries differing in their TDO-schemes are not isomorphic.

Lemma 2 *The TDO-decomposition of a finite incidence structure exists and is uniquely defined in the sense that it is independent from the choice of $S = (\mathcal{V}, \mathcal{B}, \mathbf{I})$ out of the class of incidence structures which are isomorphic to S. Moreover, the decomposition is also canonical, i. e., the ordering of classes is fixed.*

PROOF. In the description of the TDO-algorithm above, we already showed that the stated tactical decomposition exists. In order to show the independence of the resulting decomposition from the particular choice of S out of its isomorphism class we note the following: Up to duality, the basic step in the TDO-algorithm is a refinement of a partition $P^{(i)} \vdash \mathcal{V}$ to a new partition $P^{(i+1)} \prec P^{(i)}$ such that the corresponding decomposition $(P^{(i+1)}, Q^{(i)})$ is point-tactical. Such a refinement is always possible according to Lemma 1. In computing $P^{(i+1)}$, one has to ensure that this refinement is always obtained in the same manner, for example by sorting the type vectors of the points with respect to the current decomposition of blocks lexicographically. The result of the TDO-algorithm depends heavily on the chosen ordering of these type vectors. However, once a fixed algorithm for determining the refinement $P^{(i+1)}$ is chosen, the result of the TDO-algorithm is independent from the ordering of points and blocks within the original incidence geometry S. This means that the result depends only on the isomorphism type of S and not on the specific labelling of points and blocks in $S = (\mathcal{V}, \mathcal{B}, \mathbf{I})$.

It follows from Lemma 2 that the decomposition of the TDO respects the orbits of the automorphism group. This means that its classes are unions of whole orbits of the automorphism group of S. We call such a decomposition *characteristic*. A fine decomposition by the TDO may thus be exploited during the search for possible automorphisms of the geometry: only points (and blocks) of the same class can be mapped on each other. In other words: the TDO allows to colour the points and blocks in such a way that the colouring is respected under isomorphisms. This fact may aid the search for automorphisms or isomorphisms between different structures (compare [10]) and can also be used to speed up the search for a canonical form of a given incidence matrix.

Example 3 On 10 points there is a unique proper linear space. It is the affine plane of order 3 with one parallel class of lines of length 3 intersecting in an additional point (these lines are in fact 4-lines). The automorphism group has order 108. The result of the TDO-algorithm for the incidence matrix of this plane is shown in Figure 3. Heavy lines indicate the classes of the decomposition. During the refinement step, we have rearranged the classes of the refined decomposition in a lexicographically decreasing manner. This means that we first have the class of 4-lines and then the class of 3-lines. For the points, we first have the point with three 4-lines on it and then the remaining 9 points with only 1 line of length 4 and 3 lines of length 3. The TDO-schemes are shown to the right. ⋄

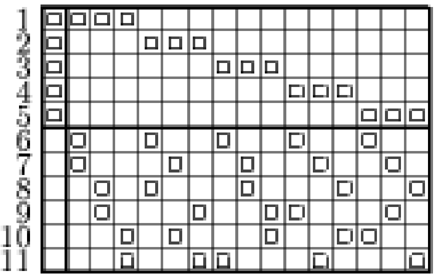


$\alpha_{i,j}$	3	9
1	3	0
9	1	3

$\beta_{i,j}$	3	9
1	1	0
9	3	3

Fig. 3. The Unique Proper Linear Space on 10 Points

Example 4 On 11 points there is again a unique proper linear space. It may be seen as the dualized complete graph on 6 points, K_6, with a parallel class of 3-lines intersecting in an additional point (in the dual space where blocks become points). It has an automorphism group of order 120 isomorphic to S_5. In the action on the 6 points of the graph, we get a transitive action of S_5 on 6 points. In Figure 4, we show the incidence geometry and the TDO-scheme together with generators for the automorphism group. Here and in the following we only show the TDO-scheme containing the $\alpha_{i,j}$ if nothing else is stated. The values $\beta_{i,j}$ can be deduced via Equation (3). On the first 5 points, we have the group S_5 acting in its usual manner, generated by a 5-cycle and a transposition: $\sigma = (1, 5, 4, 3, 2)$ and $\tau = (1, 2)$. On the points $\{6, \dots, 11\}$, we see a different action on 6 points. After subtracting 5, we get the permutations $\bar{\sigma} = (1, 3, 4, 2, 6)$ and $\bar{\tau} = (1, 6)(2, 4)(3, 5)$. Moreover, the map $\sigma \mapsto \bar{\sigma}$ and $\tau \mapsto \bar{\tau}$ defines a homomorphism between these two group actions. The two actions correspond to different subgroups S_5 of S_6. These subgroups are not conjugate in S_6 as they are of different type as permutation groups (one has



$\alpha_{i,j}$	1	15
5	1	3
6	0	5

$$Aut = \langle (1, 5, 4, 3, 2)(6, 8, 9, 7, 11), (1, 2)(6, 11)(7, 9)(8, 10) \rangle$$

Fig. 4. The Unique Proper Linear Space on 11 Points

a fixpoint, the other has none). Nevertheless, it is well known that there is an outer automorphism of S_6 mapping one group onto the other. ⋄

5 Parameters of Linear Spaces

The parameters of geometries which will be presented now differ from those of Dembowski in [7]. This is due to the fact that linear spaces do not necessarily satisfy any of the regularity conditions $(R.1)$ or $(\overline{R}.1)$. The following definitions appeared already in [3], where they were used during the construction and classification of linear spaces on up to 12 points. We repeat from there. We give all definitions for linear spaces, however, they can easily be adapted to more general incidence structures. Let $S = (\mathcal{V}, \mathcal{B})$ be a linear space.

5.1 The line type or parameters of the first kind

The *parameters of the first kind* of a geometry are the lengths of lines. Define

$$a_i \;:=\; \#\text{ lines of length } i \text{ in } S. \tag{6}$$

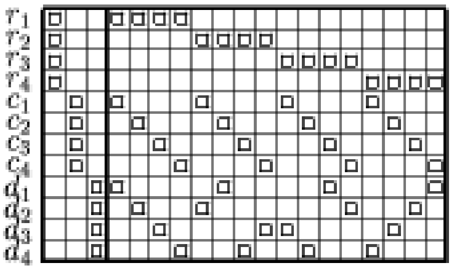
The vector $\mathbf{a} := (a_2, a_3, \ldots, a_v)$ is the *line type* of S. For convenience, the line type is often written in exponential notation, i. e., $(2^{a_2}, 3^{a_3}, \ldots, v^{a_v})$. Here, exponents 1 may be left out while terms with exponent 0 may vanish completely. Clearly, in a linear space the equation

$$\binom{v}{2} \;=\; \sum_{j=2}^{v} \binom{j}{2} a_j \tag{7}$$

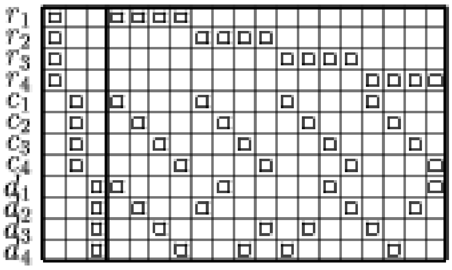
is satisfied.

Example 5 The proper linear spaces on 7, 9, 10 and 11 points (Examples 1, 2, 3, 4) have the following line types: (3^7), (3^{12}), $(3^9, 4^3)$, $(3^{15}, 5)$. ⋄

Example 6 On 12 points there are three different (non-isomorphic) proper linear spaces (cf. Figure 5). Two correspond to the Latin squares of order 4 in a way described in [2] (the automorphism groups are of order 576 and 192). The third space turns out to be the dual of the affine plane of order 9, i. e., a configuration $12_3 9_4$ admitting a parallel class of 4-lines. This space has 432 automorphisms. The first two spaces have line type $(3^{16}, 4^3)$ whilst the third has $(3^4, 4^9)$. ⋄

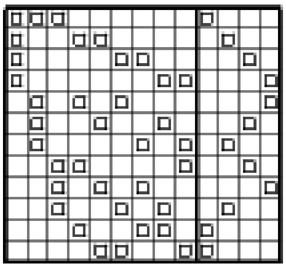


	c_1	c_2	c_3	c_4
r_1	1	2	3	4
r_2	2	1	4	3
r_3	3	4	1	2
r_4	4	3	2	1



	c_1	c_2	c_3	c_4
r_1	1	2	3	4
r_2	2	1	4	3
r_3	4	3	1	2
r_4	3	4	2	1

	3	16
12	1	4



	9	4
12	3	1

Fig. 5. The Proper Linear Spaces on 12 Points

5.2 *The point case or parameters of the second kind*

Often, considering line types suffices to distinguish between different spaces. If it does not, one may take a closer look at the parameters of the geometries. Therefore, we define *parameters of the second kind*. Let $S = (\mathcal{V}, \mathcal{B})$ be a linear space of line type $\mathbf{a} = (2^{a_2}, 3^{a_3}, \ldots, v^{a_v})$. For $p \in \mathcal{V}$ let

$$\mathbf{b}_p \ := \ ([p]_v, \ldots, [p]_3, [p]_2) \tag{8}$$

be the *point type* of p (recall that $[p]_j$ is the number of lines of length j passing through p). Sometimes, we prefer exponential notation $\mathbf{b}_p = (v^{[p]_v}, \ldots, 2^{[p]_2})$. Clearly, for any point type the equation

$$\sum_{j=2}^{v} [p]_j \, (j-1) \; = \; v-1 \tag{9}$$

holds. Now, we forget about the *particular linear space* S and instead consider the set of *all* linear spaces with a given line type $\mathbf{a}$. As each point in a linear space has a well-defined point type, we may solve (9) in all different ways to get a list of *putative* point types. However, we should solve (9) under the additional restriction

$$[p]_j \; \leq \; a_j \tag{10}$$

for $j = 2, \ldots, v$. Now, we proceed in the following way:

Let $\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_\ell$ be a complete set of solutions to (9) and (10). Denote the i-th solution by $\mathbf{b}_i = (b_{i,v}, \ldots, b_{i,2})$. For $i = 1, \ldots, \ell$, let

$$c_i \; := \; |\{ \, p \in \mathcal{V} \mid \mathbf{b}_p = \mathbf{b}_i \, \}| \tag{11}$$

be the number of points of type $\mathbf{b}_i$ in the geometry. The vector $(c_1, \ldots, c_\ell)$ is the *point type distribution* or the *point case* of the geometry. For a point type distribution, the following linear equations hold for all j with $2 \leq j \leq v$:

$$\sum_{i=1}^{\ell} c_i \, b_{i,j} \; = \; j a_j. \tag{12}$$

Apart from the obvious equation

$$\sum_{i=1}^{\ell} c_i \; = \; v \tag{13}$$

we get that for any pair of (distinct) numbers j_1 and j_2 in $\{2, \ldots, v\}$, the inequality

$$\sum_{i=1}^{\ell} c_i \cdot b_{i,j_1} \cdot b_{i,j_2} \; \leq \; a_{j_1} \cdot a_{j_2} \tag{14}$$

is satisfied. Here, equality holds if and only if every line of length j_1 meets every line of length j_2. In addition, the solution must satisfy

$$\sum_{i=1}^{\ell} c_i \cdot \binom{b_{i,j}}{2} \; \leq \; \binom{a_j}{2} \tag{15}$$

for $j = 2, \ldots, v$.

We carefully have to distinguish between *algebraically possible* and *geometrically realizable* parameter sets. There may exist a solution to Equations (12)–(15) — what we call an algebraically possible point case — which does not appear as a valid point case of a linear space S with line type $\mathbf{a}$ — what we call a geometrically realizable point case. See [3] for a deeper discussion of such realization problems. In that article, a lot of tests for geometrical realizability are discussed. They are useful for reducing the number of possible parameter sets.

In the next section, we generalise the recalculation of parameters and extend this computation to arbitrary depths. This is what we call the algebraic approach to the parameters of linear spaces. We will show examples of these calculations in the next two sections. Note that in [3], parameters of the first, second and third kind are used for the computation of linear spaces on 12 points.

6 Construction via TDO

It has already been pointed out that the TDO may be used for classification purposes. However, in this article we use the TDO in a rather different way. This new method uses TDO (or better: TDO-schemes) in a more restrictive sense, leading to a new strategy for construction. Instead of post-classifying a given set of objects we may reverse the process. Thus, we may start the construction of objects by initially computing possible TDO-schemes which the requested objects may have. We do this on a purely algebraical basis by calculating parameters in a way we have discussed it in the previous section. However, here we compute parameters of even higher kinds, i. e., also of fourth kind, of fifth kind and so on. We have to go on with this refining process of parameters as far as possible, this means, till we end up with a TDO-scheme. This parameter calculation is the first step of our algorithm.

As a second step, we try to realize the so computed schemes and obtain geometries for every single TDO-scheme. The geometries for any fixed scheme are closely related in the sense that their final parameters coincide. Usually, these lists of geometries are already very short and easy to handle. However, we may get a large list of possible parameter cases. The point is that these cases are disjoint in the sense that they contain non-isomorphic geometries. Thus, we may simply add the results of the different subcases in order to obtain complete lists of geometries.

The process of computing possible TDO-schemes can only be sketched in this section. It is a rather difficult task, in particular if parameters of higher kinds are involved. In any case, refinement of the parameters means solving diophantine systems of equations similar to those presented in the previous section. This can be done in a mostly algebraic manner. However, in order to reduce the number of possible cases we apply some tests for geometric realizability already at that stage. Refining the parameters step by step leads to a tree, the *parameter tree*. The nodes of this tree represent the parameter sets of different kinds. The line cases form the first level in the tree, the second kind of parameters the next level and so on. The initial parameters may be the number of points together with some restrictions like the exclusion of 2- and v-lines, for example. Starting out with this information, the higher parameters are computed step by step. Recursively, we first compute the possible line cases, then the point cases and so on. Eventually, we arrive at a tactical decomposition where no further refinement of parameters is possible. This decomposition is the TDO-decomposition which we have to take into account for our construction purposes.

It is important to mention that a TDO-scheme may be *discrete*, which means that all its point- and block-classes have size one. In this case, the TDO-scheme coincides with the incidence matrix and we identify the geometry and its TDO-scheme. We may require a discrete TDO-scheme to be geometrically realizable, i. e., the matrix of the TDO-scheme shall describe the incidence matrix of a valid linear space.

If a TDO-scheme is not discrete, there is the need for another program which constructs the set of possible geometries for that scheme. A priori, it is not clear whether or not a given scheme is realizable at all. In case that it is, the realization may or may not not be unique (see the following section for a lot of examples). However, our generator for geometries should produce all possible realizations up to isomorphism.

Summarising, we have two parts (cf. Figure 6):

(1) TDO-scheme calculation:
We compute all possible parameters for the requested geometries. Starting from an initial parameter situation, say the number of points or a line type in the sense of Equation (6), we calculate all possible TDO-schemes for this case. This means that we build up the parameter tree in a recursive manner. In each step, a parameter set is tried to be refined in order to obtain another parameter set of a higher kind. Thus for every line type, we compute all possible point cases, and for every point case we compute all possible parameters of third and even higher kind. Proceeding in this way, we get more and more closer to the real structure of the geometry. This is the algebraic part of the procedure. Note, however, that this part is not purely algebraic, since we consider already at that stage several tests for geometric realizability. Some of these tests are described in [3]. Sometimes, a parameter set cannot be refined any further. This

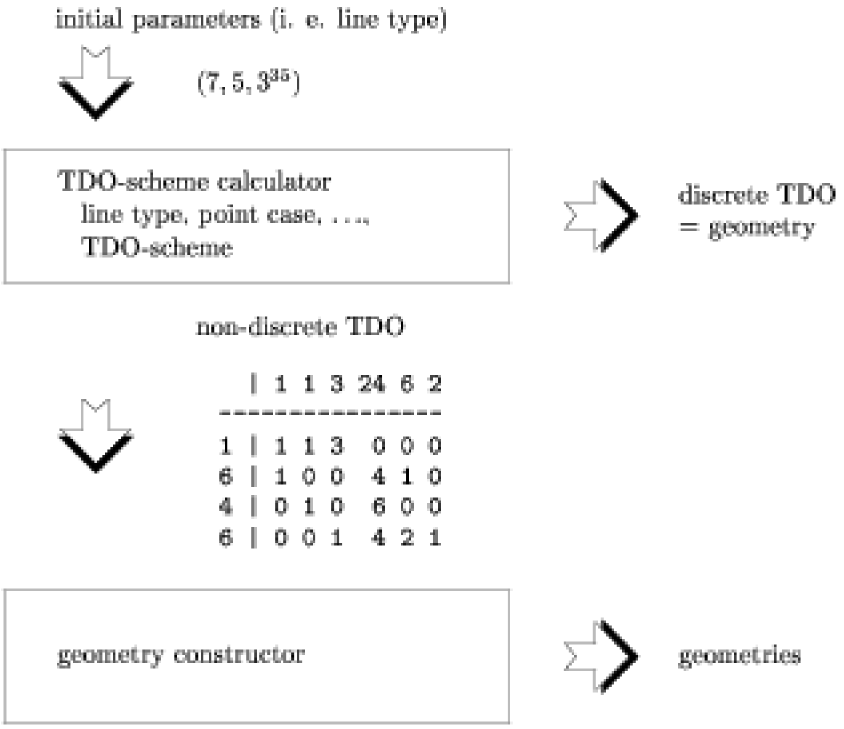


Fig. 6. The Construction Algorithm with TDO-Calculation

can happen for two different reasons. First, we may find out that there exists no parameter set of higher kind refining the current parameters. In this case we have run into a dead end and we should go back and proceed with another parameter set. On the other hand, we may have arrived at a decomposition which is tactical in both directions. A decomposition of that kind cannot be refined any further. In this case we have found a possible candidate for a TDO-scheme of the requested geometries. Such a scheme may be either discrete or not. In the first case, we are finished as we got the incidence matrix of a valid linear space (recall that we require a discrete TDO-scheme to be geometrically realizable). Otherwise, we pass the scheme to the geometry constructor.

(2) Construction of geometries:
Every non-discrete TDO-scheme of (1) may or may not be realizable. This means that it is not clear whether or not for a given TDO-scheme there exists a geometry for it. In case that a TDO-scheme can be realized, the realization may not be unique. After all, a quite general program to construct geometries defined by a given TDO-scheme is necessary. In addition, this program shall compute the possible realizations up to isomorphism. Note that for construction purposes, it is sometimes advisable to reorder the classes of the decomposition. This is due to technical reasons within the generator of geometries. The generator applies backtracking and the behaviour of algorithms of these kinds is very hard to estimate.

One remark may simplify the calculation of TDO-schemes: It is possible to omit the 2-lines in the TDO calculation as the 2-lines can always be reconstructed uniquely afterwards. Moreover, even when constructing geometries one may get along without 2-lines as well. Clearly, this remark does not apply to the construction of proper linear spaces which do not have any 2-lines. But we should keep this remark in mind in case that we plan to use the TDO-method for the construction of a different kind of species.

The depth at which a TDO-scheme shows up in the parameter tree is not fixed. In fact, it may vary from 1 to $2v$ where v is the number of points (which is less then or equal to the number of blocks by the famous theorem of de Brujin and Erdös).

We first look at the TDO at depth 1 or 2: It is possible that a line case is already a TDO. Namely, if the line case has only one type of lines then the decomposition is tactical and therefore forms a TDO. Consider, for example, the trivial linear spaces consisting of just one long line of length v or the space consisting only of 2-lines (the complete graph K_v). These two spaces have line type $(v^1) = (v)$ and $(2^{\binom{v}{2}})$ leading to TDO-schemes of size 1×1:

$$\begin{array}{c|c} & 1 \\ \hline v & 1 \end{array} \qquad \text{and} \qquad \begin{array}{c|c} & \binom{v}{2} \\ \hline v & v-1 \end{array}.$$

None of these two spaces is a proper linear space but, for example, every non-trivial Steiner-System $S(2, k, v)$ is a proper linear space on v points with a TDO-scheme of size 1×1. Thus, the 80 $S(2, 3, 15)$ are configurations $15_7 35_3$. As a matter of fact, there is no 1×1 TDO-scheme for proper linear spaces on 17 points.

Another TDO which is detected early in the parameter tree is that of a $1 \times m$ decomposition. In the corresponding linear space, all points have the same point type and this is what we call a regular linear space. For $m \geq 2$, these TDO are computed at level two in the parameter tree. To be precise, a regular linear space with parameters $(v|[p]_2, [p]_3, \dots, [p_v])$ corresponds to the following TDO-scheme (with possibly empty block classes):

$$\begin{array}{c|cccc} \alpha_{i,j} & [p]_v & \dots & [p]_3 & [p]_2 \\ \hline v & \frac{v[p]_v}{v} & \dots & \frac{3[p]_3}{v} & \frac{2[p]_2}{v} \end{array} \quad \text{or} \quad \begin{array}{c|cccc} \beta_{i,j} & [p]_v & \dots & [p]_3 & [p]_2 \\ \hline v & v & \dots & 3 & 2 \end{array}$$

Note that $[p]_v$ is either 0 or 1 and in the latter case the space happens to be the long line (which by definition is not a proper linear space).

In contrast to these cases we have the discrete TDO which are detected rather late in the calculation. Note that when refining partitions we are usually splitting more than one class at a time. Thus, even the discrete TDO-schemes do not need $2v$ steps for computation.

Example 7 The proper linear spaces on 13 points turn out to have a lot of different parameter sets. First, we list all possible TDO-schemes (cf. Figure 7). The number of geometries in these cases is 2, 0, 1, 1, 1, 2 (the second one cannot

(1)	26
13	6

(2)	5	15	1
3	0	5	1
10	2	3	0

(3)	6	8	6
1	0	0	6
12	2	2	1

(4)	4	3	12
12	1	1	3
1	4	0	0

(5)	13
13	4

(6)	3	16
12	1	4
1	3	0

Fig. 7. The TDO-Schemes for Proper Linear Spaces on 13 Points

be realized). More precisely, we have the following species:

(1) The two Steiner-Triple-Systems on 13 points.
(2) None.
(3) The dual consists of cubic graphs on 8 points joined with 4-regular graphs on 6 points. There is an additional parallel class of lines of length 6 which intersect in an additional point.
(4) Configurations 12_3 together with two different parallel classes: three lines of length 4 and four lines of length 3 forming a 4×3 grid. The 3-lines intersect in an additional point.
(5) The projective plane of order 3 or the (unique) symmetric configuration 13_4.
(6) The two Latin squares of order 4 from Example 6 again. Here, the three 4-blocks intersect in an additional point.

The line types are (3^{26}), $(3^{16}, 4^5)$, $(3^{14}, 4^6)$, $(3^{12}, 4^7)$, (4^{13}), $(3^{16}, 5^3)$. Figure 8 shows the parameter tree for the TDO-schemes. The nodes in the first level stand for line types. The numbering of these types is not continuous as there are a lot of line types which are not realizable. We do not show them here, they are simply the algebraic solutions to Equation (7) of Section 5. The nodes at level two represent point type distributions. In branches 1 and 5, the TDO-scheme is finished as we only have one point type. In all other cases, there are two different point types and the decomposition is not necessarily block-tactical. In branch 6 it is and this will be recognised in the next step. In branches 2, 3 and 4 we have another refinement of lines at level 3. Here, some 3-lines or 4-lines are divided into different classes because of the number of incidences which they have with points of different type. In level four, all resulting decompositions are tactical. Note that apart from the line type in all cases the first possible extension (refinement) of parameters is successful, indicated by the ones labelling the nodes. ◇

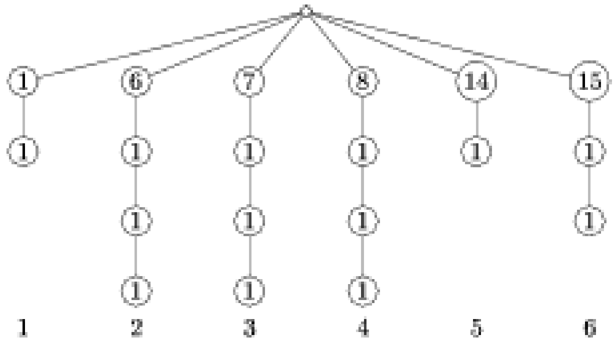


Fig. 8. The Parameter Tree on 13 Points

7 The Proper Linear Spaces on 17 Points

This section contains the results of the computation and classification of proper linear spaces on 17 points. Table 2 shows the results by line type (second column). A particular line type may or may not be realizable as a TDO or even as a geometry. Therefore, we have more line types than shown in the table (the table shows only realizable line cases, i. e., those which lead to at least one TDO). The number in the first column indicates the line type number.

Column 4 indicates the total number of TDO obtained for that particular line case. For a TDO, we have to distinguish two cases: either the TDO is discrete or not. Column 5 shows the number of discrete and non-discrete TDO. The overall number of realizations of non-discrete TDO (for a particular line case) is listed in column 6 (under 'GEO (n.d.)'). Non-discrete TDO may have multiple realizations as it happens for instance in the last row with line case $(3^{35}, 5, 7)$. There are 23 realizations of a single non-discrete TDO (we will come back to this parameter case later). But it is also possible that a non-discrete TDO does not possess a realization. For example in the preceeding row, the line case $(3^{25}, 4^{6}, 5, 6)$ leads to a non-discrete TDO which is not realizable. Interestingly, the line case $(3^{6}, 4^{8}, 5^{7})$ one row above leads to a non-discrete TDO which has a unique realization (see below).

The last column lists the total number of geometries for each line case. This number is the sum of the geometries from discrete TDO and non-discrete TDO. At the bottom of the table, the total number of geometries and TDO cases is shown.

Table 2
The Proper Linear Spaces on 17 Points

no.	line case	TDOnd-no.	# TDO = d. + n.d.	GEO (n.d.)	GEO
1	$(3^{42}\ 5)$	1	$1 = 0+1$	$157,151$	$157,151$
6	$(3^{32}\ 4^5\ 5)$	$2-6$	$1,671 = 1,666+5$	54	$1,720$
7	$(3^{30}\ 4^6\ 5)$	$7-71$	$1,280 = 1,215+65$	184	$1,399$
8	$(3^{28}\ 4^7\ 5)$	$72-84$	$125 = 112+13$	44	156
9	$(3^{26}\ 4^8\ 5)$	$85-88$	$169 = 165+4$	127	292
10	$(3^{24}\ 4^9\ 5)$	$89-105$	$111 = 94+17$	30	124
11	$(3^{22}\ 4^{10}\ 5)$	$106-117$	$37 = 25+12$	9	34
12	$(3^{20}\ 4^{11}\ 5)$	$118-119$	$3 = 1+2$	0	1
13	$(3^{18}\ 4^{12}\ 5)$	120	$1 = 0+1$	1	1
14	$(3^{16}\ 4^{13}\ 5)$	$121-124$	$4 = 0+4$	3	3
15	$(3^{14}\ 4^{14}\ 5)$	125	$1 = 0+1$	0	0
16	$(3^{12}\ 4^{15}\ 5)$	$126-128$	$3 = 0+3$	2	2
17	$(3^{10}\ 4^{16}\ 5)$	$129-130$	$2 = 0+2$	0	0
23	$(3^{32}\ 5^4)$	$131-209$	$879 = 800+79$	205	$1,005$
29	$(3^{20}\ 4^6\ 5^4)$	$210-220$	$12 = 1+11$	3	4
30	$(3^{18}\ 4^7\ 5^4)$	$221-223$	$3 = 0+3$	0	0
31	$(3^{16}\ 4^8\ 5^4)$	$224-229$	$6 = 0+6$	6	6
32	$(3^{14}\ 4^9\ 5^4)$	$230-231$	$2 = 0+2$	1	1
33	$(3^{12}\ 4^{10}\ 5^4)$	232	$1 = 0+1$	0	0
37	$(3^4\ 4^{14}\ 5^4)$	233	$1 = 0+1$	0	0
39	$(4^{16}\ 5^4)$	234	$1 = 0+1$	1	1
40	$(3^{22}\ 5^7)$	$235-237$	$3 = 0+3$	1	1
48	$(3^6\ 4^8\ 5^7)$	238	$1 = 0+1$	1	1
67	$(3^{25}\ 4^6\ 5\ 6)$	239	$1 = 0+1$	0	0
159	$(3^{35}\ 5\ 7)$	240	$1 = 0+1$	23	23
			$4,319 = 4079+240$	$157,846$	$161,925$

Finally, it should be remarked that for construction purposes, the non-discrete TDO are numbered sequentially, starting with 1 for the TDO of line case $(3^{42}, 5)$. Altogether, we have 240 non-discrete TDO. Column 3 indicates the non-discrete TDO numbers within each line case.

Figure 9 shows the parameter tree of proper linear spaces on 17 points. Only the subtree of non-discrete TDO is shown here. Due to lack of space, we can only show parts of the tree. Line cases 7 and 13–39 are removed from the figure. The nodes at level i correspond to sets of parameters of depth i of possible geometries. Leaves are TDO-schemes and may occur at arbitrary level. The non-discrete TDO numbers are shown in the line at the bottom.

Let us begin with an example of a TDO and consider line case no $1 \doteq (3^{42}, 5)$ in the first row of Table 2. In the parameter tree, this case is the leftmost branch of the tree. Using the TDO calculator, we get a unique TDO at depth 3:

$$\begin{array}{l} \text{line type:} \\ (5,\ 3^{42}) \end{array} \;\rightarrow\; \begin{array}{l} \text{point type distribution:} \\ \ \ 5 \times (1,6) \\ 12 \times (0,8) \end{array} \;\simeq\; \begin{array}{r|rr} \alpha_{i,j} & 1 & 42 \\ \hline 5 & 1 & 6 \\ 12 & 0 & 8 \end{array}$$

$$\rightarrow\; \begin{array}{r|rrr} \beta_{i,j} & 1 & 30 & 12 \\ \hline 5 & 5 & 1 & 0 \\ 12 & 0 & 2 & 3 \end{array} \;\simeq\; \begin{array}{r|rrr} \alpha_{i,j} & 1 & 30 & 12 \\ \hline 5 & 1 & 6 & 0 \\ 12 & 0 & 5 & 3 \end{array} \;\leadsto\; \begin{array}{r|rrr} \alpha_{i,j} & 1 & 30 & 12 \\ \hline 12 & 0 & 5 & 3 \\ 5 & 1 & 6 & 0 \end{array}$$

Note that in step two (after computing the point type distribution) we have a point-tactical decomposition which is *not* block-tactical. For the generation process, it proves to be useful to exchange the two point classes in the TDO as indicated after the $\leadsto$-symbol. From Table 2, we find that there are altogether 157,151 realizations. We only show the first one (with trivial automorphism group) in Figure 10. The incidence matrix is decomposed with respect to the TDO and shown in its canonical form.

The geometries of this parameter case consist of a configuration 12_3 overlaid by a regular graph of degree 5. This graph has a partition of its edges into 5 parallel classes. This means that there are 5 disjoint classes of edges and every vertex is contained in exactly one edge of every class. Each parallel class of lines intersects in an additional point, and these extra points are joined by a 5-line. Thus, we get a space on 12+5=17 points. It is well known that there are 229 configurations 12_3 (see for example [2] or [4]). The number of 5-regular graphs on 12 points is 7849 (cf. M. Meringer [15]). 7848 of them are connected and one is the union of two copies of the unique 5-regular graph on 6 points, the complete graph on 6 points K_6. The configuration 12_3 determines

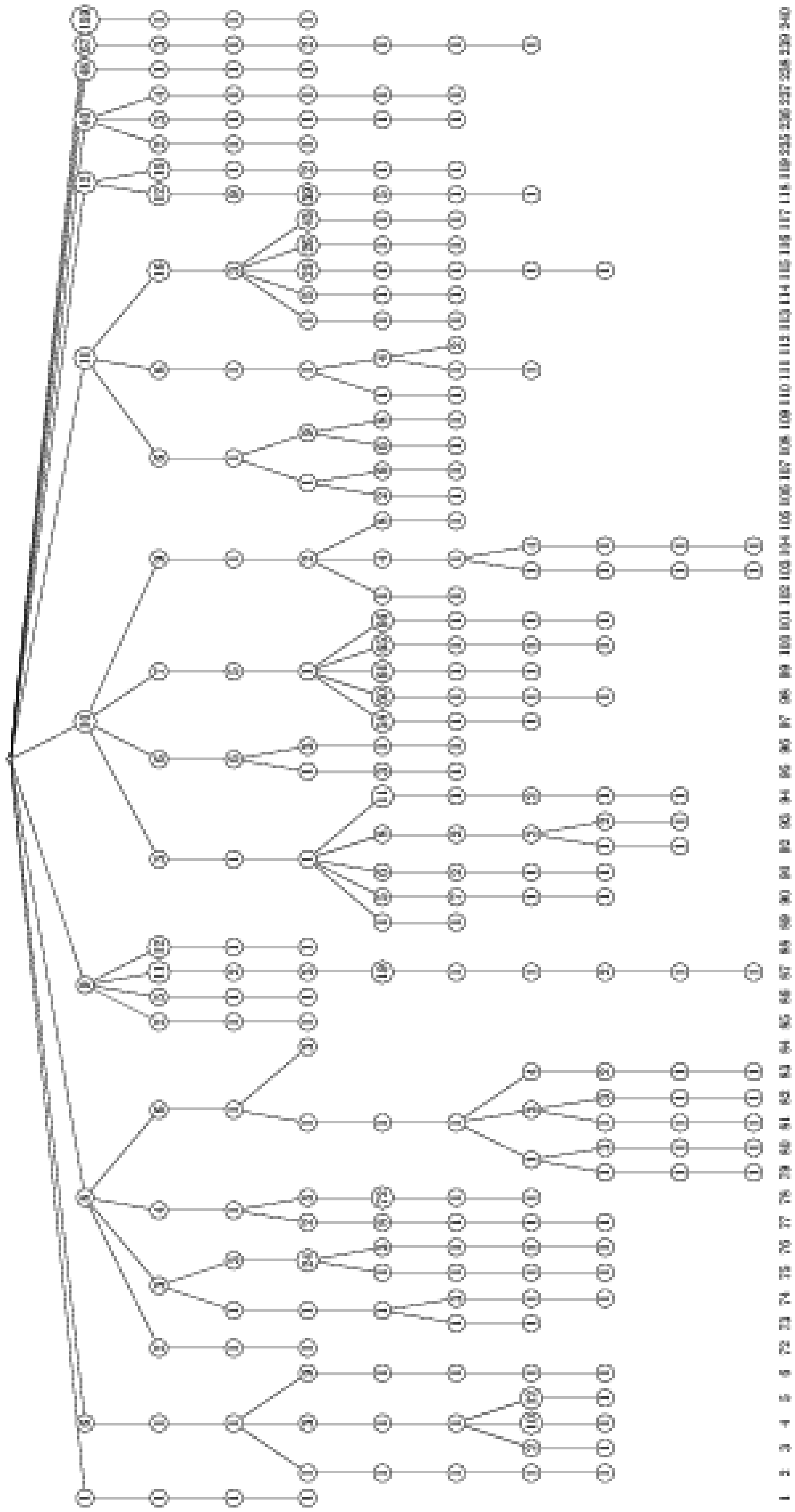


Fig. 9. The Parameter Tree of Non-Discrete TDO-Schemes (Shortened)

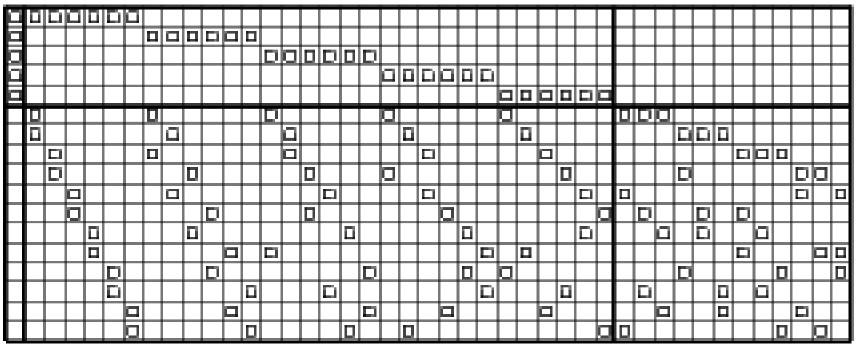


Fig. 10. One of the 157,151 Geometries of Line-Type $(3^{42}, 5)$

the graph uniquely as its configuration graph (see the remarks to this point in Section 2). However, the parallel classes are not unique causing the high number of isomorphism types of geometries in this case.

In order to give a typical example of one case in the list, consider line case no 159 $\hat{=} (3^{35}, 5, 7)$ (the rightmost branch in Figure 9). There exists a unique TDO-scheme possessing 23 realizations:

	1	1	3	24	6	2
1	1	1	3	0	0	0
6	1	0	0	4	1	0
4	0	1	0	6	0	0
6	0	0	1	4	2	1

A uniquely realizable line case is no 48$\hat{=}(3^6, 4^8, 5^7)$. We get the following TDO-scheme from the TDO calculator:

	1	6	8	6
3	1	2	0	2
12	0	2	2	1
2	1	0	4	0

The unique realization is shown in Figure 11 (in its canonical form). This geometry has an automorphism group of order 48. It is generated by the following permutations (of points or rows):

(1 2)(5 8)(6 12)(7 13)(10 14)(11 15)(16 17),
(1 3 2)(4 6 12)(5 10 13)(7 14 8)(9 11 15),
(4 5)(8 9)(12 14)(13 15)(16 17).

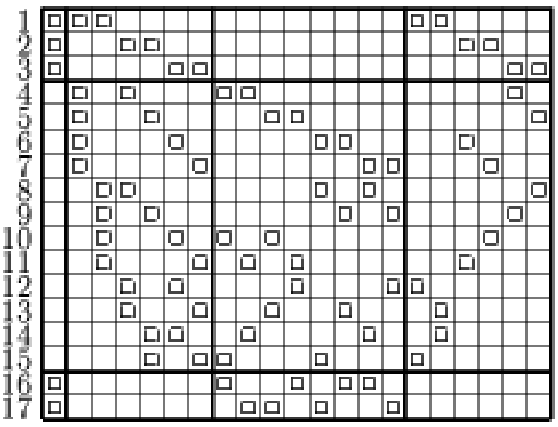


Fig. 11. The Unique Geometry of Line-Type $(3^6, 4^8, 5^7)$

Fairly interesting is a subcase of line case $23 \triangleq (3^{32}, 5^4)$. This line case leads to 79 non-discrete TDO. The first is no 131

	4	32
16	1	6
1	4	0

and has 23 realizations. On the first 16 points, we have the linear spaces of Franek, Mathon and Rosa [9]. They are extended by an additional point in which all four 4-lines intersect.

It seems instructive to analyse also non-discrete TDO which have no realization. We display two examples. TDO 238 is rather fine (Figure 12) whereas TDO 233 has only few point and block classes (Figure 13).

In order to study an example of a discrete TDO-scheme, consider the one shown in Figure 14 with 17 point- and 33 block-classes. Clearly, the TDO coincides with the incidence matrix. This TDO was deduced from line case $(3^{22}, 4^{10}, 5)$. It came up at depth 10. The path labelling the cases at each step is 11.8.1.1.3.1.1.3.1.1. Here the i-th number stands for the case at depth i, e. g. the first 11 stands for the line case 11 (cf. Table 2). The partitions of points and blocks at intermediate levels are not indicated. These parameters can be recalculated by applying TDO-algorithm on the incidence matrix. In the first step, we simply count the column sums yielding 1×5, 10×4 and 22×3 coinciding with the line type. After that, we count the point types and

	1	1	1	2	2	1	2	2	2	2	2	2	2	2	2	2	2	2	1
1	1	1	1	0	0	0	2	0	0	0	0	0	0	0	0	0	0	0	0
2	1	0	0	1	0	0	0	0	1	0	0	1	1	0	0	0	1	0	0
2	1	0	0	0	1	0	0	1	0	0	1	0	0	1	0	1	0	0	0
1	1	0	0	0	0	1	0	0	0	2	0	0	0	0	2	0	0	0	0
2	0	1	0	1	1	0	0	1	1	1	0	0	0	0	0	0	0	0	0
2	0	1	0	0	0	0	0	0	0	0	1	1	1	1	1	0	0	1	0
2	0	0	1	1	0	0	0	0	0	1	1	0	0	0	0	1	1	1	0
1	0	0	1	0	0	1	0	2	0	0	0	2	0	0	0	0	0	0	1
2	0	0	0	1	1	0	1	0	0	0	0	0	1	0	1	1	0	0	1
2	0	0	0	0	1	1	1	0	1	0	0	0	0	1	0	0	1	1	0

Fig. 12. A non-realizable TDO for line case $(3^{25}, 4^6, 5, 6)$

$\alpha_{i,j}$	4	12	2	3	1
6	2	2	0	1	0
8	1	3	1	0	0
3	0	4	0	1	1

or

$\beta_{i,j}$	4	12	2	3	1
6	3	1	0	2	0
8	2	2	4	0	0
3	0	1	0	1	3

Fig. 13. A non-realizable TDO for line case $(3^4, 4^{14}, 5^4)$

get the follwing point type distribution forming the parameters at depth 2:

$\alpha_{i,j}$	1	10	22
1	1	4	0
2	1	2	3
2	1	0	6
4	0	4	2
8	0	2	5

The next step is another refinement of blocks. By simple counting, we get the

	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
1	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	1	0	0	0	0	1	1	0	0	0	0	1	0	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	1	0	0	0	0	0	0	1	1	0	0	0	1	1	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	1	0	1	0	1	0	0	1	0	1	0	0	0	0
1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	1	0	1	0	1	1	0	1	0	0	0	0	0
1	0	1	0	0	0	1	0	0	1	1	0	0	0	0	0	0	0	0	0	0	1	1	0	0	0	0	0	0	0	0	0	0	0
1	0	0	1	0	0	0	1	1	0	1	0	0	0	0	0	0	0	0	0	1	0	0	0	0	1	0	0	0	0	0	0	0	0
1	0	0	0	1	0	1	0	1	0	0	1	0	0	0	0	0	0	0	1	0	0	0	0	1	0	0	0	0	0	0	0	0	0
1	0	0	0	0	1	0	1	0	1	0	1	0	0	0	0	0	0	1	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0
1	0	1	0	0	0	0	0	1	0	0	0	1	0	0	0	0	0	1	0	0	0	0	0	0	0	1	0	0	0	1	0	0	1
1	0	0	1	0	0	0	0	0	1	0	0	1	0	0	0	0	0	0	1	0	0	0	0	0	0	0	1	0	0	0	1	1	0
1	0	0	0	1	0	0	1	0	0	0	0	0	1	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	1	1	1	0	0
1	0	0	0	0	1	1	0	0	0	0	0	0	0	1	0	0	0	0	0	1	0	0	0	0	0	0	0	1	0	1	0	1	0
1	0	0	1	0	0	0	0	0	0	0	1	0	1	0	1	0	0	0	0	0	0	1	0	0	0	0	0	1	0	0	0	0	1
1	0	0	0	1	0	0	0	0	0	1	0	0	0	1	0	1	0	0	0	0	0	0	1	0	0	0	1	0	0	0	0	0	1
1	0	1	0	0	0	0	0	0	0	0	1	0	0	0	0	1	1	0	0	0	0	0	0	1	0	0	0	0	1	0	0	1	0
1	0	0	0	0	1	0	0	0	0	1	0	0	0	0	1	0	1	0	0	0	0	0	0	1	0	1	0	0	0	0	1	0	0

Fig. 14. A Discrete TDO for Line-Type $(3^{22}, 4^{10}, 5)$

following block-tactical decomposition:

$\beta_{i,j}$	1	4	4	2	6	8	4	4
1	1	1	0	0	0	0	0	0
2	2	0	1	0	1	0	0	0
2	1	0	0	0	0	1	1	0
4	0	1	2	2	0	1	0	0
8	0	2	1	2	2	1	2	3

After 7 further iterations one arrives at the discrete TDO of Figure 14 .

Finally, we would like to point out that the results presented here may be obtained electronically from our homepage for this article. The address is

`http://www.mathe2.uni-bayreuth.de/betten/PUB/pub_proper17.html`

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