Linear Spaces with at Most 12 Points

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ABSTRACT

The 28,872,973 linear spaces on 12 points are constructed. The parameters of the geometries play an important role. In order to make generation easy, we construct possible parameter sets for geometries first (purely algebraically). Afterwards, the corresponding geometries are tried to construct. We define line types, point types, point cases and also refined line types. These are the first three steps of a general decomposition according to the parameters which we call TDO. The depth of parameter precalculation can be varied, thereby obtaining a handy tool to react in a flexible way to different grades of difficulty of the problem. © (Year) John Wiley & Sons, Inc.

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1. INTRODUCTION

A linear space P on v points is a collection $\mathcal{B} = \{B_1, \ldots, B_b\}$, of subsets of P called blocks (or lines) such that every block has at least two points and each pair of points is contained in exactly one block.

If a point $p \in P$ is contained in a block $B \in \mathcal{B}$ we also say that "p lies on B," or "B passes through p," or "p and B are incident." The number of lines passing through a fixed point p is called the *degree*, denoted by $[p]$. The number of lines of length j passing through p is called the *j-degree*, $[p]_j$ for short. A system of blocks which satisfies the modified axioms such that each line has at least two points and any pair of points is contained in *at most one* block is called a *prelinear space*.

Usually, such a set of points together with a distinguished set of subsets is called *incidence geometry*. A priori, all points are equal and so there is a notion of equivalence (called isomorphism) which comes from exchanging points. To be precise, two such geometries are isomorphic if and only if one can be obtained from the other by a bijective map of the points which preserves incidences. The isomorphisms of a space with itself form a group, the *automorphism group*. When we speak about equal or different linear spaces we mean isomorphic or non-isomorphic ones, respectively.

Sometimes, we also need the notion of the *dual geometry*. It can be obtained by reversing the roles of points and blocks and keeping the relation of incidence. The dual of the dual is always isomorphic to the original space. The dual of a linear space is only a prelinear space, in general. Often one is interested in obtaining a complete set of (different) linear spaces on a given number of points. Let LIN(v) be the number of elements in such a list. For our purposes, it is important to verify that such a list of linear spaces on v points is both *complete* and *irredundant*, that is, no space is missing and all spaces in the list are pairwise different.

The linear spaces on very few points are easily listed: on the empty set, there is one space consisting of no blocks. On a single point, there is again one space with no blocks. On a two-point set there is one linear space formed by a single 2-line joining both points. On three points there are two different spaces: The first one has a line of length 3 joining all the points. The other one has three 2-lines forming a triangle. On four points there are three spaces: one 4-line, a 3-line and three 2-lines, and six 2-lines. So, LIN$(v) = 1, 1, 1, 2, 3$ for $0 \leq v \leq 4$.

Often one visualizes a linear space by drawing the blocks as lines and the points as nodes in the plane. Sometimes it is necessary not just to draw straight lines but to allow also arcs and circles for the blocks. In a lot of cases, various 2-lines are omitted from the drawing because they are redundant (one can always reconstruct the two-lines if they are left out). Figure 1 shows all linear spaces on five points.

An *incidence matrix* of a geometry (linear space) is the 0, 1-matrix $M = (m_{ij})$ of size $v \times b$ with $m_{ij} = 1$ if and only if point no. i lies on line j, that is, p_i and B_j are incident. For aesthetic reasons however, we replace ones by little boxes in the drawings of this article. An empty square stands for a 0, that is, a nonincidence. Figure 2 shows incidence matrices of all linear spaces on five points. Incidence matrices are a handy tool for putting linear spaces on a computer. But one is faced with the problem that there can exist different incidence matrices for one and the same space. Namely, permuting rows and columns of a given incidence matrix does not change the space but often leads to other incidence matrices. It is therefore

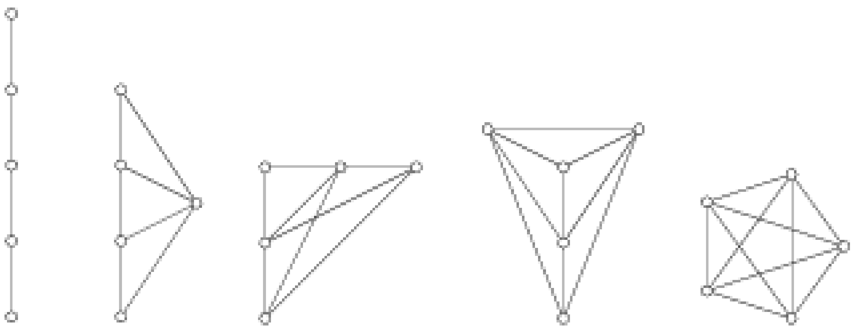


FIG. 1. The Linear Spaces on Five Points

useful to introduce so-called *canonical incidence matrices*. Such a matrix is defined to be the lexicographically least representative among all the incidence matrices of a given space. The canonical form is unique and there exist algorithms to compute it.

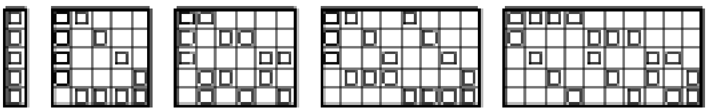


FIG. 2. Canonical Incidence Matrices for the Linear Spaces on Five Points

The number, LIN(v), of linear spaces on v points is Sloane's Sequence no. 271 [17] (encyclopedia number A001200 [18]):

v	0	1	2	3	4	5	6	7	8	9	10	11	12
LIN(v)	1	1	1	2	3	5	10	24	69	384	5,250	232,929	28,872,973

1.1 Historical Notes

The systematic enumeration (and construction) of linear spaces has been started by Doyen [11] in 1967. He constructed linear spaces on up to 9 points. It took more than 20 years until D. Betten and D. Glynn continued in 1990 and computed the 5250 linear spaces on 10 points (independently). The next step was the computation of linear spaces on 11 points: D. Betten and M. Braun [5] invented the "TDO" method which is mainly an algorithm for computing a good invariant useful for a preclassification of the geometries. *Without the use of isomorphism tests*, they were able to give a lower bound for the number of linear spaces on 11 points. As a matter of fact, there were only six spaces more, namely 232,929, as computed independently by Ch. Pietsch [16] and D. Betten together with C. Kuhse. The book of L. Batten and A. Beutelspacher [1] contains a lot of drawings of linear spaces on small point sets.

Sometimes, linear spaces with certain properties are studied. For example, a linear space is called *proper* if it does not contain 2-lines. The proper linear spaces on up to 15 points are constructed by Brouwer [7]. The proper linear spaces on 16 points are also known (cf. G. Heathcote [14]). Recently, the current authors determined the proper linear spaces on 17 points [4].

Camina and Mischke [9] look predominantly at linear spaces where all lines have the same length and which have an automorphism group transitive on lines (and imprimitive on points). Gropp [13] studies $(r, 1)$-designs which are linear spaces whose points all have degree r.

Another important class of linear spaces is the following. A linear space is called *regular* if the j-degree of a point depends only on j (and not on the point). In other words: the j-degrees are all equal in the space. In this case, restriction to the lines of length j gives a configuration for each j. In [2], the current authors determine all regular linear spaces on up to 16 points (with only few exceptional cases).

The sequence, LIN(v), is contained twice in the CRC Handbook of Combinatorial Designs [10]. There is a general section about "pairwise balanced designs as linear spaces" which is due to H.-D. Gronau, R.C. Mullin and Ch. Pietsch [12] and a section about classical geometries by A. Beutelspacher which contains a short passage on linear spaces [6].

1.2 The Plan of This Article

The general strategy for this article is the following: In order to compute linear spaces we start with the parameters of geometries. By parameters we mean for instance the distribution of lines of different length in the geometry (see Section 2.) But we will go further and consider also finer parameters. We call them parameters of higher kind and they can be either point or line parameters. Point types, for instance, specify the number of lines of any given length passing through a fixed point. These point types may occur with different multiplicities in the space and the distribution of point types is called point case (see Section 3.) We will also introduce refined line cases which describe how the points of different type are located on the lines of different length. This will be done using a lot of examples in Section 4.

Precalculating parameters up to a certain step proves to be useful with respect to some important points: First, the generation of spaces becomes easier if much about the parameters is known. Strictly speaking, generation means the process of computing certain 0/1-matrices which serve as incidence matrices for spaces of that type. Generation is done by taking into account several constraints: usually, the row and column sums for these matrices are prescribed. Sometimes, one has even more, namely there may be a finer partitioning of rows and columns and the number of incidences is known within the areas of this decomposition. Moreover, one may always assume that within each part of such a partition of points (or blocks) all the rows (or columns, respectively) are sorted lexicographically decreasing. Proceeding in this way, one may reduce the number of possible matrices considerably in a lot of cases. A second major benefit from using decompositions is that canonical forms can often be easier computed using them. The classes of a decomposition give a sort of precoloring of elements and these colors have to be respected during the search for canonical forms. A coloring is good if it has a lot of different colors and in this case the search for the canonical form is simplified. Third, it is a priori clear that spaces with different parameter sets are not isomorphic. Therefore, approximation via the parameters will break the problem up into a lot of small pieces which can be handled independently and more easily.

We should also mention possible drawbacks of this method: Clearly, there is a certain amount of overhead in computing parameters. The computation may take a while and not each parameter case is realizable as a linear space. Therefore, we will develop various tests for realizability in the sequel. These tests are able to reduce the amount of cases considerably. It is also advisable to allow variation in the depth of parameter precalculation. Some cases are easier to handle than others with respect to generation purposes. Thus, only few parameters should be computed in general. But there exist hard cases and it showed to be useful to apply deeper parameter calculations for them. So parameter calculation provides a handy tool for adapting to different grades of difficulty of the problem. One is able to react in a flexible way by choosing an appropriate depth of parameter precalculation.

Section 5. displays our results. For $v = 7$–12, the number of linear spaces is shown according to the line type.

It should be remarked that the method of parameter precalculation really anticipates the so-called TDO process (cf. [5] or [3]) during classification of geometries. This means that for any geometry belonging to a fixed parameter set of kind one, two or three, computing TDO reveals exactly these parameters in the first three steps. Sometimes these parameters already coincide with the TDO (we will see an example of this case in Section 4.2). Otherwise, we get at least an approximation of the TDO which is still useful. This is an approximation from the top, so the TDO is always a refinement of the decompositions obtained from the first few parameter cases. The parameter precalculation can be extended to arbitrary depths in principle. For instance, it is possible to compute the complete TDO on a purely algebraical basis without handling with incidence matrices. The TDO is the final stage of all parameter precalculations. There is no further refinement possible due to the fact that it is tactical. The TDO-decomposition is *characteristic* in the sense that the automorphism group respects the classes. However, it may occur that the orbits of the automorphism group are indeed strictly finer than the TDO.

Recently, the method of parameter precalculation has been applied in its full generality going as far as computing TDO-parameters in all cases. This means that there was no fixed limit in the depth of parameters and that the program was able to do a quite general step of parameter refinement which generalizes the methods presented here. However, for the beginning it seems to be of great help to start with some explicit parameter cases before going further. The current authors determined the proper linear spaces on 17 points using this more general approach [4].

2. PARAMETERS OF THE FIRST KIND

Let (P, B) be a linear space on v points. Define

$$a_i := \#\,\text{lines of length } i \text{ in } (P, B). \tag{1}$$

The vector $\mathbf{a} := (a_2, a_3, \ldots, a_v)$ is called the *line type* of the space (P, B). Line types are also called *parameters of the first kind* of the geometry. Often, it is convenient to denote line types in exponential notation, that is, $(2^{a_2}, 3^{a_3}, \ldots, v^{a_v})$. Exponents 1 may not be explicitly written, terms with exponent 0 are left out. For sake of brevity, one may even omit the 2-lines. They are redundant and one can compute their number from the rest as we will see soon.

Example: The linear spaces on five points have the following line cases:

$$(5),\ (2^4, 4),\ (2^4, 3^2),\ (2^7, 3),\ (2^{10}).$$

One can visualize the distinction between lines of different length in the incidence matrix by introducing bold lines as in Figure 3. Thus we get a partitioning of the block set into classes. We will also partition the point set in a like manner. We call such partitionings of points and/or blocks *decompositions*. They can be either point- or block-tactical or even both. This means that the number of incidences of one representative of a given point- (block-) class with all elements of a given block- (point-) class is independent of the choice of that particular representative. The decompositions which we are working with are not always tactical. As our partitions come from structural data of the space, the group of automorphisms will respect them. In other words, there is no automorphism φ sending a line of one type to a line of another type (and no point of one type may be mapped onto a point of another type). In [3], such a partitioning is called a *characteristic decomposition*. Throughout this article, bold lines in incidence matrices indicate characteristic decompositions.

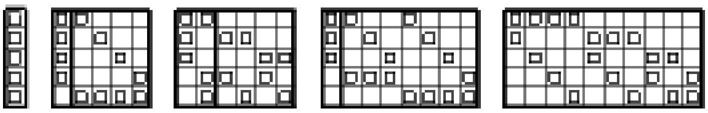


FIG. 3. Linear Spaces on Five Points with Different Length Lines Separated

So, the linear spaces on five points can be distinguished by their line type. This is no longer true for the spaces on six points (cf. Fig. 4). Table I shows the (number of) linear spaces on 6 points by their line type. There are ten geometries, the line type (3^2) is realized twice (no. 7 and 8 in Fig. 4). Clearly, the two geometries are nonisomorphic as the first one has a point of intersection of the two 3-lines whereas the other one does not have such a point.

TABLE I. Linear Spaces on 6 Points by Line Type

Line Case	#Geo	Line Case	#Geo
1: (6)	1	6: $(2^3\ 3^4)$	1
2: $(2^5\ 5)$	1	7: $(2^6\ 3^3)$	1
3: $(2^6\ 3\ 4)$	1	8: $(2^9\ 3^2)$	2
4: $(2^9\ 4)$	1	9: $(2^{12}\ 3)$	1
5: (3^5)	0	10: (2^{15})	1
Total:			10

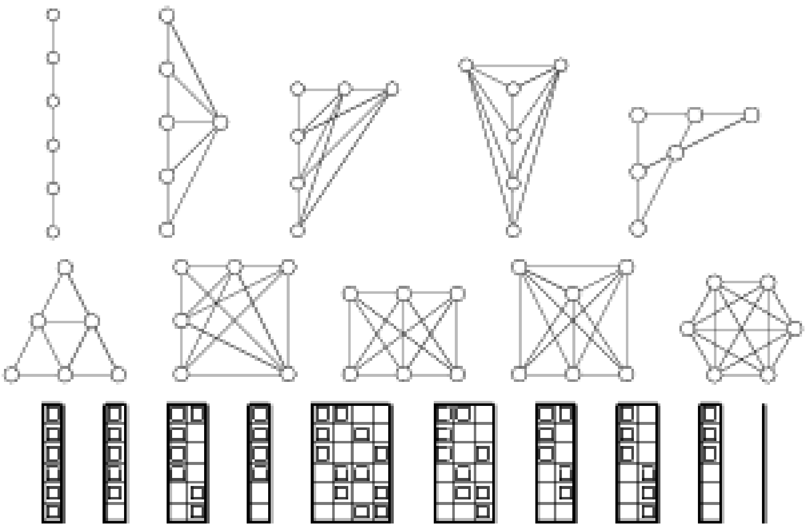


FIG. 4. The Linear Spaces on Six Points, Incidence Matrices by Leaving Out The Two-Lines

In the following, we will be concerned with a purely algebraic task of precomputing line types. Afterwards, some geometric plausibility considerations are made which reduce the amount of cases to consider drastically.

As each i-line joins $\binom{i}{2}$ pairs of points and as each pair of points in P is joined by exactly one line we get

$$\sum_{i=2}^{v} a_i \binom{i}{2} = \binom{v}{2}. \tag{2}$$

2.1 The de Brujin / Erdős Test

Not all line types which fulfil (2) can be realized. For instance, we saw already that there is no linear space on six points with five 3-lines. It is a challenge to precompute putative parameter sets in such a way that the probability that these sets are realizable as linear spaces is high.

The following important theorem is a first step in that direction.

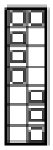
Theorem 2.1 (De Brujin, Erdős [8]). *Let $P = (V, B)$ be a linear space. If B is different from the line of length v then $b \geq v$ holds.*

The theorem of de Brujin and Erdős gives even more but we need only this part. We deduce that there cannot be a linear space on six points with only five lines.

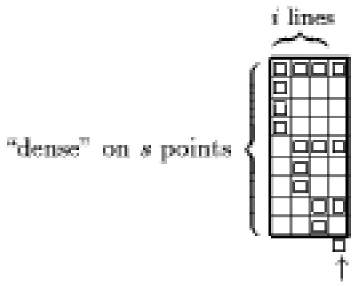
There are a lot of proofs of the theorem of de Brujin and Erdős some of which came up recently. Probably the most beautiful one is due to Conway. See Metsch [15] or van Lint and Wilson [19] for more.

2.2 The Minimum Breadth Test

Let us consider another nonrealizable line type: for example, on seven points it is impossible to have a linear space with two 4-lines, one 3-line and six 2-lines though $6 \cdot \binom{2}{2} + \binom{3}{2} + 2 \cdot \binom{4}{2} = 21 = \binom{7}{2}$. The two 4-lines must intersect because there are only seven points. But then, no 3-line is possible. In terms of the incidence matrix:



One can generalize this to an easy test for filtering out possible line types. Starting with the longest lines, one tries to place them as "close" as possible, that is, with the smallest number of points involved. Assume one has placed i lines "dense" and one is going to place another line with only few new points needed. Then, only i incidences can be made with the first s points as it can be seen for instance in the following example. Here, three 4-lines are placed on nine points which is the minimal number of points possible. The incidence matrix does not show a linear space because of its first and fifth row. But for our test it is only important to know which lines intersect therefore we proceed trying to place a fourth 4-line. This is impossible with only nine points because there can be at most three intersections with the three other lines.



More generally, having placed i lines on a minimum of s points (dense packing), one needs $k - i$ *new* points when placing an additional k-line. If $s + k - i > v$ this is impossible. This criterion works best if one starts with placing long lines first. Here is the algorithm — we specify it in some formal language which is close to real programming languages.

Algorithm: verification of line types via minimum breadth test.
    input: a line case $\mathbf{a} = (a_2, a_3, \ldots, a_v)$
        for a linear space on v points.
    output: TRUE if the line case makes sense
        (passes "min-breadth" test), FALSE otherwise
    int i, k, ℓ, s, m;
    $i := 0$;

```
s := 0;
for k := v down to 2
    for ℓ := 1 to a_k                                   // loop invariant:
                                   // we have placed i lines on at least s points
        m := max(k − i, 0);        // we need at least m new points
        s := s + m;
        if (s > v)
            return FALSE;          // line type cannot be realized
        i := i + 1;
    end;
end;
return TRUE;                                   // line type seems to be OK
```

Note that there are line types which pass this test but which cannot be realized. For example, the line types $(3^3, 4^6)$, $(2^3, 3^2, 4^6)$, $(2^6, 3, 4^6)$ or $(2^9, 4^6)$ on ten points are not ruled out. Also $(3, 4^7)$ and $(2^3, 4^7)$ are still valid though they cannot be realized. Some of these cases can be excluded due to the de Brujin / Erdős test but we will now present another test which is able to eliminate all these line cases, too.

2.3 The Maximal Flag Test

Let us determine the maximal number of incidences which fit into a rectangular matrix of dimension $i \times j$ say. Putting the geometrical conditions of a linear space aside for the moment one starts in the following way: place the incidences "tightly", that is, start from the top left position and fill the incidences row by row into the matrix. Consider the line cases with six 4-lines on ten points, for example. One gets the situation of Figure 5 after dualizing.

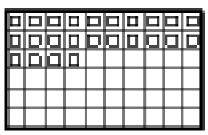


FIG. 5. Tight Packing of Incidences

We find that $4\binom{3}{2} + 6\binom{2}{2} = 12 + 6 = 18$ pairs of rows are joined whereas in the geometry only $\binom{6}{2} = 15$ pairs of rows are possible (any placement of the incidences covers at least as many two-subsets as the packing in the example). So, there is no 6×10 incidence matrix with row sum four for a dualized space on ten points. To be slightly more general, we conclude that there is no geometry which has a 6×10 incidence matrix with more than 22 incidences (this is because 22 incidences give 2 columns of weight 3 and 8 columns of weight 2 and $2\binom{3}{2} + 8\binom{2}{2} = 6 + 8 = 14 \leq 15 = \binom{6}{2}$ whilst for 23 incidences we get $3\binom{3}{2} + 7\binom{2}{2} = 9 + 7 = 16 > 15$).

More formally, we claim that the following is true:

Lemma 2.2. *For a geometry with n incidences on i points with j blocks to exist,*

$$n \leq \text{Maxfit}[i][j] := \min\left(\max_m f(m,j) \leq \binom{i}{2},\ \max_m f(m,i) \leq \binom{j}{2}\right) \tag{3}$$

must be satisfied. Here, $f(m,j) := \binom{\lfloor m/j \rfloor}{2} \cdot j + (m \bmod j) \cdot \lfloor m/j \rfloor$ *and* $a \bmod b$ *for* $a, b \in \mathbb{Z}$ *is the unique integer* c *with* $0 \leq c < b$ *and* $a \equiv c \bmod b$.

Proof. The tight packing of incidences indicated in Figure 5 gives us a lower bound on the number of pairs of points (corresponding to rows) which must be joined in an incidence geometry with n points. This can be deduced by noticing that the tight packing is obtained by repeated application of the following three kinds of operations:

1. sliding an incidence into a higher box of the same column if that higher field and all fields in-between are empty,
2. permuting columns
3. raising a box (incidence) from the end of a long column to the end of a shorter column (cf. Fig. 6).

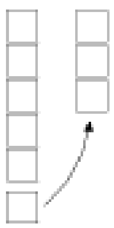


FIG. 6. Preparing for the Maxfit Test

The number of pairs which are joined may be expressed in terms of column sums of the incidence matrices. Assume we have x_k incidences in the k-th column for $k = 1, \ldots, j$. The function $P(x_1, \ldots, x_j) = \sum_{k=1}^{j} \binom{x_k}{2}$ counts these pairs. Clearly, operation 1 does not change the x_k. Operation 2 simply permutes these values and therefore P is not changed. The third operation always reduces the value of this function. Therefore, P decreases weakly during the succession of operations of type 1-3. This shows that we can obtain a lower bound from the tight packing which has column sums $y_k = \lfloor n/j \rfloor + 1$ for $k = 1, \ldots, n \bmod j$ and $y_k = \lfloor n/j \rfloor$ for $k = n \bmod j + 1, \ldots, j$. Therefore,

$$P(y_1, \ldots, y_j) = f(n,j) \leq P(x_1, \ldots, x_j) \tag{4}$$

where $x_1, \ldots, x_j$ are the column sums of any incidence geometry with n incidences in an $i \times j$ grid. Clearly,

$$P(x_1, \ldots, x_j) \leq \binom{i}{2} \tag{5}$$

holds and (4) together with (5) imply $f(n,j) \leq \binom{i}{2}$. Applying this test to both, the geometry and its dual, we get the statement of the lemma. □

The first few Maxfit numbers are shown in Table II. The lower triangle of the matrix is not shown, it is the transpose of the upper triangular part (the symmetry in i and j comes from application of this test for both, the geometry and its dual).

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
1	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
2		3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26
3			6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28
4				9	10	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31
5					12	14	15	17	18	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35
6						16	18	19	21	22	24	25	27	28	30	31	32	33	34	35	36	37	38	39	40
7							21	22	24	25	27	28	30	31	33	34	36	37	39	40	42	43	44	45	46
8								25	27	29	30	32	33	35	36	38	39	41	42	44	45	47	48	50	51
9									30	32	34	36	37	39	40	42	43	45	46	48	49	51	52	54	55
10										35	37	39	41	43	45	46	48	49	51	52	54	55	57	58	60
11											40	42	44	46	48	50	52	54	56	57	59	60	62	63	65
12												46	48	50	52	54	56	58	60	62	64	66	67	69	70
13													52	54	56	58	60	62	64	66	68	70	72	74	76
14														57	60	62	64	66	68	70	72	74	76	78	80
15															63	66	68	71	73	75	77	79	81	83	85
16																70	72	75	77	80	82	84	86	88	90
17																	76	79	81	84	86	89	91	93	95
18																		83	85	88	90	93	95	98	100
19																			90	92	95	97	100	102	105
20																				97	100	102	105	107	110
21																					105	107	110	112	115
22																						112	115	117	120
23																							119	122	125
24																								127	130
25																									135

TABLE II. Maxfit Numbers for $i, j \leq 25$

Finally, we would like to point out that there are also the true maxfit-numbers, that is, maxfit$[i][j]$ is the largest n such that there exists a geometry with n incidences in an $i \times j$ field. Clearly, maxfit$[i][j] \leq$ Maxfit$[i][j]$ but the deviation of the upper bound is hard to compute as determining maxfit numbers involves a severe construction problem. Geometries whose numbers of incidences attain the true maxfit numbers deserve special interest.

3. PARAMETERS OF THE SECOND KIND

Let us come back to the two linear spaces on six points with line type $(2^9, 3^2)$ (the seventh and eighth geometries in Fig. 4). In order to distinguish the two spaces we were looking at the point degrees. In the first geometry there was one point with two 3-lines and one 2-line, one point with five 2-lines and there were four points with one 3-line and three 2-lines. In the other geometry each point had one 3-line and three 2-lines. Therefore, we are led to the following refinement of parameters.

3.1 Point Types and Point Type Distributions

Let (P, B) be a linear space on v points with line type $\mathbf{a} = (2^{a_2}, \ldots, v^{a_v})$. For fixed $p \in P$ we define

$$\mathbf{b}_p := ([p]_v, \ldots, [p]_3, [p]_2) \tag{6}$$

the *point type* of p. Usually we prefer exponential notation $\mathbf{b}_p = (v^{[p]_v}, \ldots, 2^{[p]_2})$. The vector $\mathbf{b}_p$ is called the *point type* of p. The multiset of vectors of point types $\{\mathbf{b}_p \mid p \in P\}$ is the *point type distribution* or *point case* of the geometry. The line case together with the point case form the *parameters of the second kind* of the geometry.

For the two $(2^9, 3^2)$, we get the following point types

$$\left\{ \begin{array}{l} 1 \times (3^2, 2) \\ 4 \times (3, 2^3) \\ 1 \times (2^5) \end{array} \right. \quad \text{and } 6 \times (3, 2^3).$$

Visualized in the incidence matrix we have:

Let us now switch over from the type of a particular point p to the set of all possible point types in linear spaces with a given line type. Therefore, we remove the reference to the point p from our notation and write $\mathbf{b} = (b_v, \ldots, b_3, b_2)$. The following important question immediately arises: What are the necessary conditions for such a vector of non-negative integers to be a valid point type in a linear space with line type $\mathbf{a} = (a_2, a_3, \ldots, a_v)$? Clearly,

$$b_j \leq a_j \tag{7}$$

must be satisfied for each $j = 2, \ldots, v$. As each point in a linear space is joined to each other point and as each line of length j joins a fixed point to $j-1$ other points we get

$$\sum_{j=2}^{v} b_j(j-1) = v - 1. \tag{8}$$

Let now $\mathbf{b}_i = (b_{i,v}, \ldots, b_{i,2})$ run through all solutions to (7) and (8) with $i = 1, \ldots, k$. Set

$$c_i = \#\,\text{points of type } \mathbf{b}_i \text{ in } (P, B) \quad \text{for } 1 \leq i \leq k. \tag{9}$$

The vector $\mathbf{c} = (c_1, c_2, \ldots, c_k)$ is the *point type distribution*. We are now going to compute point types and point type distributions for linear spaces of a given line type.

3.2 Counting Incidences

Fix a line case $\mathbf{a} = (a_2, a_3, \ldots, a_v)$ and assume that $\mathbf{b}_1, \ldots, \mathbf{b}_k$ are all possible point types according to (7) and (8). One forms a $k \times v-1$ matrix $B = (b_{i,j})$ with the point types written in its rows. To compute possible point type distributions (or point cases) one considers a blocking scheme of the incidence matrix of possible geometries (cf. Fig.7).

	$\overbrace{a_v \times v\text{-lines}}$	$\dots$	$\overbrace{a_3 \times 3\text{-lines}}$	$\overbrace{a_2 \times 2\text{-lines}}$
$c_1 \times \mathbf{b}_1$ {	$\leftarrow b_{1,v} \rightarrow$	$\dots$	$\leftarrow b_{1,3} \rightarrow$	$\leftarrow b_{1,2} \rightarrow$
$c_2 \times \mathbf{b}_2$ {	$\leftarrow b_{2,v} \rightarrow$	$\dots$	$\leftarrow b_{2,3} \rightarrow$	$\leftarrow b_{2,2} \rightarrow$
$\vdots$	$\vdots$		$\vdots$	$\vdots$
$c_k \times \mathbf{b}_k$ {	$\leftarrow b_{k,v} \rightarrow$	$\dots$	$\leftarrow b_{k,3} \rightarrow$	$\leftarrow b_{k,2} \rightarrow$
	$\sum = va_v$	$\dots$	$\sum = 3a_3$	$\sum = 2a_2$

FIG. 7. choosing points of different types

The blocking of the matrix is induced by the line-type (vertical stripes) and the point cases (horizontal stripes). In order to determine possible point distributions $(c_1, c_2, \dots, c_k)$ we proceed in the following way: Counting the incidences in the strip of j-lines in two ways leads to the following system of Diophantine equations:

$$\sum_{i=1}^{k} c_i b_{i,j} \;=\; j a_j \quad \text{for } 2 \le j \le v. \tag{10}$$

Clearly, the sum of all c_i is fixed:

$$\sum_{i=1}^{k} c_i \;=\; v. \tag{11}$$

The solutions to (10) and (11) give all possible point cases for geometries $(P, \mathcal{B})$ with line type $\mathbf{a}$. Note that there might exist "algebraically possible" point cases which are not realizable. In order to get more necessary conditions we apply the tests of Sections 2.2 and 2.3.

1. (the minimum-breadth test of Section 2.2) Consider the dualized geometry of the set of j-lines of $(P, \mathcal{B})$ – assume $a_j > 0$. These geometries are prelinear spaces on a_j points and have c_i lines of length $b_{i,j}$ for $1 \le i \le k$. The minimum breadth test must be satisfied for all such geometries. Again, this test is best applied after reordering the lines to obtain decreasing line lengths.
2. (the maximal flag test of Section 2.3) The intersection of points of type $\mathbf{b}_i$ with lines of length j form a $c_i \times a_j$ submatrix of the incidence matrix. Assume that $c_i > 0$ and $a_j > 0$ to avoid trivial cases. We apply the maximal flag

test of Section 2.3 to each such submatrix. The point case is valid only if the condition is fulfilled in all cases.

3. (the maximal flag test for combinations of squares) It is also possible to apply the previous test to combinations of those submatrices. For example, one could stack two such matrices of the same column upon each other and apply the maxflag-test with $c_i b_{i,j} + c_{i'} b_{i',j}$ incidences in a $c_i + c_{i'}$ by a_j grid.

See Section 3.5 for a worked-out example where all these tests are applied.

3.3 Counting Intersections

In order to reduce the number of putative parameter sets further we may apply another test. Let us choose two different columns belonging to j_1-lines and j_2-lines, say, in Figure 7 and assume that $a_{j_1} > 0$ and $a_{j_2} > 0$. We count the number of intersections *between* lines of these different sorts. As each two lines intersect in at most one point, there can be no more than $a_{j_1} \cdot a_{j_2}$ intersections between these two sorts of lines. Each point of type $\mathbf{b}_i$ lies in the intersection of $b_{i,j_1} \cdot b_{i,j_2}$ such pairs of lines and thus the following inequality must be satisfied

$$\sum_{i=1}^{k} c_i \cdot b_{i,j_1} \cdot b_{i,j_2} \;\leq\; a_{j_1} \cdot a_{j_2} \quad \text{for } j_1, j_2 \in \{2, \ldots, v\},\ j_1 \neq j_2. \tag{12}$$

3.4 The j-Degree Test

For the next test, let j be fixed and consider lines of length j $(2 \leq j \leq v)$. Fix a point p with maximal j-degree. Consider the set of points $q \neq p$ covered by the pencil of j-lines through p. More formally, we set

$$X \;=\; \{q \in V \setminus \{p\} \mid \exists B \in \mathcal{B} : |B| = j,\ p \in B,\ q \in B\}. \tag{13}$$

Clearly,

$$X \cup \{p\} \;\subseteq\; \{q \in V \mid [q]_j > 0\} =: Y \tag{14}$$

holds. Counting yields (cf. Fig. 8)

$$|X| \;=\; [p]_j \cdot (j-1) \quad \text{and} \quad |Y| \;=\; \sum_{i:\, b_{i,j} > 0} c_i. \tag{15}$$

(14) and (15) together imply

$$[p]_j \cdot (j-1) < \sum_{i:\, b_{i,j} > 0} c_i \tag{16}$$

and this gives another necessary condition for second kind parameter sets.

3.5 A Worked-Out Example

We finish this section on second kind parameters with an example on 8 points showing how to combine parameter calculation, application of various kinds of tests and construction seen so far.

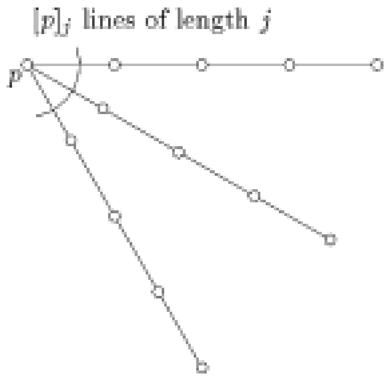


FIG. 8. The j-Degree Test

Assume we want to construct all linear spaces on eight points with line type $(2^{16}, 3^4)$. We start with the point types. According to (8), we solve

$$2\,b_3 + b_2 = 7$$

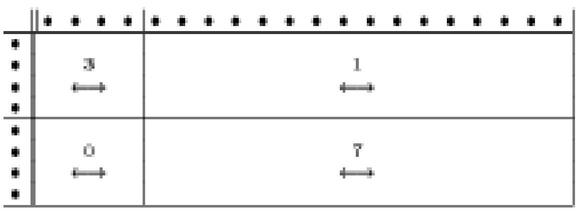
and find the solutions $\mathbf{b} = (b_3, b_2) = (3,1)$, $(2,3)$, $(1,5)$ and $(0,7)$. In order to compute point type distributions, we solve (10) together with (11):

$$\begin{array}{rcrcrcrcr} 3\,c_1 & + & 2\,c_2 & + & c_3 & & & = & 12 \\ c_1 & + & 3\,c_2 & + & 5\,c_3 & + & 7\,c_4 & = & 32 \\ c_1 & + & c_2 & + & c_3 & + & c_4 & = & 8 \end{array}$$

Starting with the solution $c = (c_1, c_2, c_3, c_4) = (4, 0, 0, 4)$ we get the following parameters of our linear spaces

	4	16
4	3	1
4	0	7

which should serve as a short way to describe the incidence matrix with the following indicated row-sums:



(each bullet stands for a row or a column of the incidence matrix). But what about realizability of this parameter set? Looking at the topmost box in the first column we find a geometry with $4 \cdot 3 = 12$ incidences in a 4×4 rectangle. But Maxfit[4][4] = 9 so this is impossible. We conclude that $c_1 < 4$. Note that the minimum-breadth test is able to rule out this case, too.

The cases $c_1 = 3$ and $c_1 = 2$ also lead to a violation of the maxfit condition, so $c_1 \leq 1$. We continue with $c = (1, 4, 1, 2)$. Thus we get the following parameters:

	4	16
1	3	1
4	2	3
1	1	5
2	0	7

The top left rectangle is now fine but still we cannot realize this parameter set. The problem lies in the two topmost rectangles in the first column: combining them, we get an incidence-matrix with $1 \cdot 3 + 4 \cdot 2 = 11$ incidences in a 5×4 field. But Maxfit[4][5] = 10 so this is impossible.

We try $c = (1, 3, 3, 1)$ with

	4	16
1	3	1
3	2	3
3	1	5
1	0	7

This leads to our first realization (cf. Fig. 9).

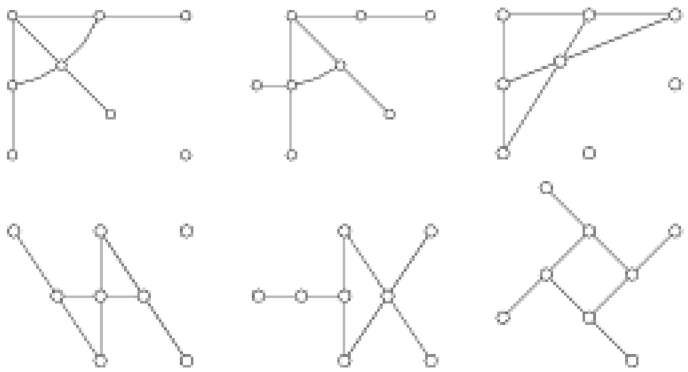


FIG. 9. The 6 Example Spaces

The next case is $c = (1, 2, 5, 0)$ which also possesses a realization.

	4	16
1	3	1
2	2	3
5	1	5

From now on, $c_1 = 0$. The next case is $c = (0, 6, 0, 2)$. In

	4	16
6	2	3
2	0	7

we find the dual of the complete graph K_4 in the top left square. The corresponding geometry exists.

The distribution $c = (0, 5, 2, 1)$ gives another uniquely realizable parameter case (geometry no. 4).

With $c = (0, 4, 4, 0)$ we get a scheme which is realizable by two different linear spaces (no 5 and 6). The first one has a 3-line whose points all have $[p]_3 = 2$. In the other one, each 3-line has two points with $[p]_3 = 2$ and one with $[p]_3 = 1$. So, even though the second kind parameters of these two spaces coincide, a closer look at the lines shows a difference (cf. Fig. 10). This is the motivation for introducing parameters of even higher kind, see the following section.

FIG. 10. Linear Spaces on Eight Points with Same Second Kind Parameters, Refinement of the Line Type

4. PARAMETERS OF THE THIRD KIND

Let B be a line of length j in a linear space $P = (V, \mathcal{B})$ on v points. Let $\mathbf{b}_1, \ldots, \mathbf{b}_k$ be the point types in P. Define

$$d_i^B \; := \; \# \text{ of points of type } \mathbf{b}_i \text{ on } B \tag{17}$$

for $1 \leq i \leq k$. The vector

$$\mathbf{d}_B = (d_1^B, d_2^B, \ldots, d_k^B) \tag{18}$$

is called the *refined line type* of B in P.

Each line B of P has a refined line type $\mathbf{d}_B$. The multiset of refined line types $\{\mathbf{d}_B | B \in \mathcal{B}\}$ is called *refined line type distribution*. We will also call them *parameters of the third kind* of the geometry.

Following the general scheme of this article, we are now going to precompute third kind parameters of possible geometries. Therefore, we forget about the particular space P and consider all possible line types in spaces with given first and second kind parameters. Assume that $\mathbf{a} = (a_2, \ldots, a_v)$ is a fixed line type and $\mathbf{b}_1, \ldots, \mathbf{b}_k$ are the point types each occuring with multiplicity c_i. Define

$$d_{j,i} \; := \; \# \text{ of points of type } \mathbf{b}_i \text{ on a line of length } j. \tag{19}$$

The vector

$$\mathbf{d}_j = (d_{j,1}, d_{j,2}, \ldots, d_{j,k}) \tag{20}$$

is a refined line type of j-lines. It is our task to compute all refined line types and to choose them with appropriate multiplicities. These selections will form our third kind parameters of the geometries.

4.1 Refined Line Types

Assume that a line case $\mathbf{a}$ and a point case $(c_1, \ldots, c_k)$ corresponding to point types $\mathbf{b}_1, \ldots, \mathbf{b}_k$ are given. In order to compute possible types of j-lines (with $a_j > 0$ in the line type) we have to solve the equation

$$\sum_{i=1}^{k} d_{j,i} \;=\; j \tag{21}$$

under the additional restrictions

$$d_{j,i} \;\leq\; c_i \quad \text{for all } i = 1, \ldots, k. \tag{22}$$

Assume that there are ℓ_j solutions

$$\mathbf{d}_{j,1}, \mathbf{d}_{j,2}, \ldots, \mathbf{d}_{j,\ell_j},$$

and write

$$\mathbf{d}_{j,u} = (d_{j,u,1}, \ldots, d_{j,u,k}) \;\; \text{for } u \leq \ell_j.$$

Let $e_{j,h}$ be the number of (j-)lines of type $\mathbf{d}_{j,h}$ in the space. The vector

$$\mathbf{e} \;=\; (\, e_{v,1}, e_{v,2}, \ldots, e_{v,\ell_v}; \;\ldots\;;\; e_{3,1}, e_{3,2}, \ldots, e_{3,\ell_3};\; e_{2,1}, e_{2,2}, \ldots, e_{2,\ell_2}\,) \tag{23}$$

is the *refined line type distribution* or *refined line case* of the geometry. To be a little bit more precise, the line type, the point types, the point type distribution, the refined line types and the refined line type distribution altogether form the third kind parameters of the geometry. Clearly, realizability is still an important topic and nonrealizable parameter sets should again be recognized and eliminated as soon as possible.

Note that the computation of refined line types gives nothing new if there is only one point type in the point type distribution. Namely, in this case the refined line types are unique and coincide with the original line types given by the length of lines. This is what we call a TDO-case: the second kind parameters already describe a tactical decomposition which is the same as it would show up when TDO-classifying geometries of that type. It has been mentioned in the introduction that a TDO cannot be refined any further. Note that the TDO-cases appearing at level two are exactly the parameter cases of regular linear spaces (in the sense of [2]).

Clearly,

$$\sum_{h=1}^{\ell_j} e_{j,h} = a_j \quad \text{for } j = 2, 3, \ldots, v \tag{24}$$

must be satisfied. Balancing incidences within the rows belonging to a fixed point type $\mathbf{b}_i$ and the columns belonging to j-lines leads to the following equations. We call them type-1 equations (cf. Fig. 11).

$$\sum_{u=1}^{\ell_j} d_{j,u,i} e_{j,u} \;=\; c_i \, b_{i,j} \quad \text{(type-1)} \tag{25}$$

for $1 \leq i \leq k$ with $c_i > 0$, and for $2 \leq j \leq v$ with $a_j > 0$. Next, we have to ensure that each pair of points is joined exactly once. Fix an index i with $1 \leq i \leq k$ and

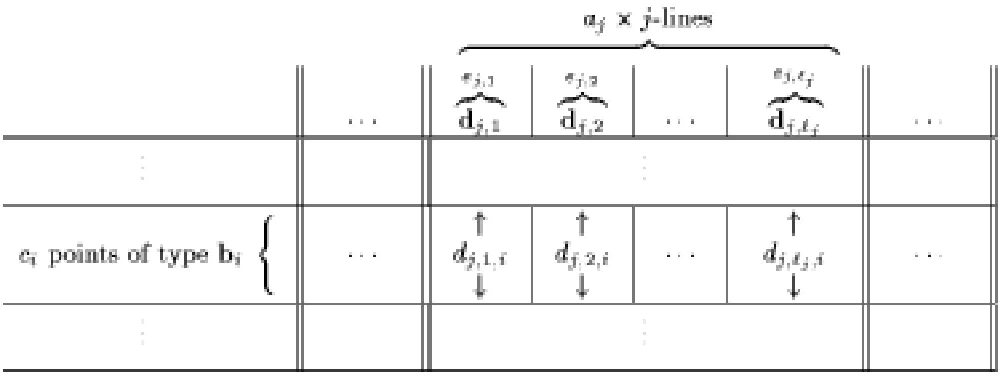


FIG. 11. choosing refined line types

$c_i > 0$. All the pairs of points of type $\mathbf{b}_i$ are joined if and only if the following equation of type 2 holds:

$$\sum_{j=2}^{v}\sum_{u=1}^{\ell_j}\binom{d_{j,u,i}}{2}e_{j,u} = \binom{c_i}{2} \text{ (type-2)}. \tag{26}$$

Finally we consider points of different type. Fix $i_1 \neq i_2$ such that $c_{i_1} > 0$ and $c_{i_2} > 0$ We get equations of type 3:

$$\sum_{j=2}^{v}\sum_{u=1}^{\ell_j} d_{j,u,i_1} \cdot d_{j,u,i_2} \cdot e_{j,u} = c_{i_1} \cdot c_{i_2} \quad \text{(type-3)} \tag{27}$$

for $1 \leq i_1, i_2 \leq k$ and $i_1 \neq i_2$. This type of equations is dual to (12). Here we have in fact equality since the points and blocks form a linear space.

4.2 The Example on Eight Points Again

Let us come back to the example of linear spaces on 8 points with line type $(2^{16}, 3^4)$. Assume we are in the last point case:

$$\begin{array}{c|cc} & 4 & 16 \\ \hline 4 & 2 & 3 \\ 4 & 1 & 5 \end{array} \tag{28}$$

Solving (21), we get the following refined line types of 3-lines and 2-lines

$$\mathbf{d}_{3,1} = (3,0),\ \mathbf{d}_{3,2} = (2,1),\ \mathbf{d}_{3,3} = (1,2),\ \mathbf{d}_{3,4} = (0,3),$$
$$\mathbf{d}_{2,1} = (2,0),\ \mathbf{d}_{2,2} = (1,1),\ \mathbf{d}_{2,3} = (0,2).$$

We are looking for solutions

$$\mathbf{e} = (e_{3,1}, e_{3,2}, e_{3,3}, e_{3,4}, e_{2,1}, e_{2,2}, e_{2,3})$$

of the following systems of equations. The type-1 equations and (24) give for the 3-lines

$$\begin{array}{rcrcrcrcl} e_{3,1} & + & e_{3,2} & + & e_{3,3} & + & e_{3,4} & = & 4 \\ 3e_{3,1} & + & 2e_{3,2} & + & e_{3,3} & & & = & 8 \\ & & e_{3,2} & + & 2e_{3,3} & + & 3e_{3,4} & = & 4 \end{array} \tag{29}$$

and for 2-lines

$$\begin{array}{rcrcrcl} e_{2,1} & + & e_{2,2} & + & e_{2,3} & = & 16 \\ 2e_{2,1} & + & e_{2,2} & & & = & 12 \\ & & e_{2,2} & + & 2e_{2,3} & = & 20 \end{array} \tag{30}$$

The equations of type 2 and 3 are:

$$\begin{array}{rcrcrcrcrcrcrcl} 3e_{3,1} & + & e_{3,2} & & & & & + & e_{2,1} & & & & & = & 6 \\ & & & & e_{3,3} & + & 3e_{3,4} & & & & & + & e_{2,3} & = & 6 \\ & & 2e_{3,2} & + & 2e_{3,3} & & & & & + & e_{2,2} & & & = & 16 \end{array} \tag{31}$$

We solve these equations with the two vectors

$$\mathbf{e}_1 = (1, 2, 1, 0, 1, 10, 5) \;\text{ and }\; \mathbf{e}_2 = (0, 4, 0, 0, 2, 8, 6).$$

They form the two different refined line type distributions of the two spaces of Figure 10 — the 2-lines were left out in the figure. This means that we can refine (28) in exactly two different ways. We obtain the following schemes (here, the numbers inside the scheme stand for column sums)

$$\begin{array}{c|cccccc} & 1 & 2 & 1 & 1 & 10 & 5 \\ \hline 4 & 3 & 2 & 1 & 2 & 1 & 0 \\ 4 & 0 & 1 & 2 & 0 & 1 & 2 \end{array} \quad \text{and} \quad \begin{array}{c|cccc} & 4 & 2 & 8 & 6 \\ \hline 4 & 2 & 2 & 1 & 0 \\ 4 & 1 & 0 & 1 & 2 \end{array} \tag{32}$$

Figure 12 shows the TDO decompositions of the two example spaces. The first scheme in (32) is already very close to its TDO whereas the second one in fact coincides with its TDO decomposition. The automorphism group of the first space has order 4 and is generated by (1 2)(5 6) and (7 8) (labeling points or rows from the top downwards in the incidence matrix). The second space has a group of order 8 generated by (1 2)(3 4)(6 7) and (1 3 4 2)(5 6 8 7). These two spaces correspond to the last two spaces of Figure 9.

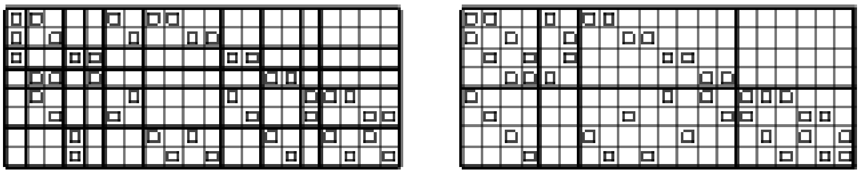
5. RESULTS

5.1 Linear Spaces by Line Types

Tables IV to X display the numbers of linear spaces on 7–12 points. We list only realizable line cases and show the number of geometries within each case. Additionally, we indicate the computing time. In order to save space, only running times of 2 or more minutes are shown.

All computations were made on a DEC AlphaStation 600 with 400 MHz CPU clocks. At the end of the tables, the total number of geometries and the the overall running time are given.

All linear spaces on ≤ 11 points were computed using only second kind parameters. The LIN(12)-computation uses different parameter depths. Table III shows



	1	2	1	1	2	4	2	2	1	4
2	1	1	0	0	1	2	0	0	0	0
1	1	0	1	1	0	0	2	0	0	0
1	0	2	0	1	0	0	0	2	0	0
2	0	1	0	0	1	0	1	0	1	2
2	0	0	1	0	0	2	0	1	0	2

	4	2	8	6
4	2	1	2	0
4	1	0	2	3

FIG. 12. The TDO Decompositions and TDO-Schemes of the Two Example Spaces

the parameter depth which has been chosen for each line type. The numbers 2 and 3 stand for second and third kind parameters, respectively, whereas a star (*) indicates that a special program was used (namely, the program written for [2]).

Note that we do not show the 2-lines in the line type. This is for reasons of space and it is possible to recompute their number using (2). Empty parentheses stand for K_v, the complete graph on v points.

The results displayed in this section can also be obtained via Internet. We maintain two copies of this page, one at the Journal's home page and one at the author's. The addresses are

`http://www.emba.uvm.edu/~jcd/reports/335/pub_lin12.html`
`http://www.mathe2.uni-bayreuth.de/betten/PUB/pub_lin12.html`

In addition to this article, the latter of these pages also contains incidence matrices for the linear spaces. One file for each line case of a linear space on v points is provided. The coding of the files is explained on the above mentioned page. Moreover, the files are compressed using the program gzip. The total amount of storage needed for the linear spaces on 11 points is only 1.1 MB whereas the linear spaces on 12 points need 117 MB of disk space. This means that on the average each linear space is coded with between 4 and 5 bytes, which is amazingly short. (The credit goes to the authors of the program gzip!)

Finally, we would like to mention the generation rate. The linear spaces on 10, 11 and 12 points were constructed at a rate of 350, 228 and 312 objects per second, respectively. However, the actual rate of generation within the individual line cases may differ from these values considerably. Moreover, in the case that parameters of depth 2 or 3 were used, the generation rate inside the line case is just the average over all the subcases resulting from refinement of parameters.

TABLE III. Different Parameter Depths for the LIN(12) Computation

Line Cases	Parameter Depth
$(12) - (4^3)$	2
$(4^2) - (3^{13}, 4^2)$	3
$(3^{12}, 4^2) - (4^2)$	2
$(4) - (3^{14}, 4)$	3
$(3^{13}, 4) - (3^{21})$	2
(3^{20})	*
(3^{19})	2
$(3^{18}) - (3^{17})$	3
(3^{16})	3, * for configurations $12_4\,16_3$
$(3^{15}) - (3^{12})$	3
$(3^{11}) - ()$	2

TABLE IV. The Linear Spaces on 7–9 Points

$v = 7$, line case	#	$v = 8$, line case	#	$v = 8$ line case	#	$v = 9$ line case	#	$v = 9$ line case	#
(7)	1	(8)	1	(3^4)	6	$(3^6\ 5)$	3	$(3^6\ 4)$	31
(6)	1	(7)	1	(3^3)	4	$(3^5\ 5)$	3	$(3^5\ 4)$	32
$(3\ 5)$	1	$(3\ 6)$	1	(3^2)	2	$(3^4\ 5)$	6	$(3^4\ 4)$	26
(5)	1	(6)	1	(3)	1	$(3^3\ 5)$	5	$(3^3\ 4)$	12
(4^2)	1	$(4\ 5)$	1	$()$	1	$(3^2\ 5)$	4	$(3^2\ 4)$	6
$(3^3\ 4)$	1	$(3^3\ 5)$	1	total:	69	$(3\ 5)$	2	$(3\ 4)$	2
$(3^2\ 4)$	1	$(3^2\ 5)$	1	time:	0 sec	(5)	1	(4)	1
$(3\ 4)$	2	$(3\ 5)$	2			$(3^4\ 4^3)$	1	(3^{12})	1
(4)	1	(5)	1	$v = 9$,		$(3^3\ 4^3)$	1	(3^{11})	1
(3^7)	1	$(3^3\ 4^2)$	1	line case		$(3^2\ 4^3)$	2	(3^{10})	4
(3^6)	1	$(3^2\ 4^2)$	1	(9)	1	$(3\ 4^3)$	1	(3^9)	12
(3^5)	2	$(3\ 4^2)$	1	(8)	1	(4^3)	1	(3^8)	31
(3^4)	3	(4^2)	2	$(3\ 7)$	1	$(3^7\ 4^2)$	1	(3^7)	41
(3^3)	3	$(3^6\ 4)$	2	(7)	1	$(3^6\ 4^2)$	3	(3^6)	34
(3^2)	2	$(3^5\ 4)$	2	$(4\ 6)$	1	$(3^5\ 4^2)$	9	(3^5)	19
(3)	1	$(3^4\ 4)$	6	$(3^3\ 6)$	1	$(3^4\ 4^2)$	12	(3^4)	11
$()$	1	$(3^3\ 4)$	5	$(3^2\ 6)$	1	$(3^3\ 4^2)$	12	(3^3)	5
total:	24	$(3^2\ 4)$	4	$(3\ 6)$	2	$(3^2\ 4^2)$	6	(3^2)	2
time:	0 sec	$(3\ 4)$	2	(6)	1	$(3\ 4^2)$	4	(3)	1
		(4)	1	(5^2)	1	(4^2)	2	$()$	1
		(3^8)	1	$(3^3\ 4\ 5)$	1	$(3^8\ 4)$	3	total:	384
		(3^7)	4	$(3^2\ 4\ 5)$	1	$(3^7\ 4)$	13	time:	0 sec
		(3^6)	6	$(3\ 4\ 5)$	1				
		(3^5)	7	$(4\ 5)$	2				

TABLE V. The Linear Spaces on 10 Points

$v = 10$ line case	#	$v = 10$ line case	#	$v = 10$ line case	#	$v = 10$ line case	#	$v = 10$ line case	#
(10)	1	$(3^3\ 5^2)$	1	$(3^5\ 5)$	38	$(3\ 4^3)$	5	$(3^3\ 4)$	22
(9)	1	$(3^2\ 5^2)$	1	$(3^4\ 5)$	26	(4^3)	3	$(3^2\ 4)$	7
$(3\ 8)$	1	$(3\ 5^2)$	1	$(3^3\ 5)$	12	$(3^{10}\ 4^2)$	2	$(3\ 4)$	2
(8)	1	(5^2)	2	$(3^2\ 5)$	6	$(3^9\ 4^2)$	6	(4)	1
$(4\ 7)$	1	$(3^4\ 4^2\ 5)$	2	$(3\ 5)$	2	$(3^8\ 4^2)$	47	(3^{13})	2
$(3^3\ 7)$	1	$(3^3\ 4^2\ 5)$	2	(5)	1	$(3^7\ 4^2)$	158	(3^{12})	28
$(3^2\ 7)$	1	$(3^2\ 4^2\ 5)$	3	(4^5)	1	$(3^6\ 4^2)$	272	(3^{11})	119
$(3\ 7)$	2	$(3\ 4^2\ 5)$	1	$(3^6\ 4^4)$	1	$(3^5\ 4^2)$	235	(3^{10})	332
(7)	1	$(4^2\ 5)$	1	$(3^5\ 4^4)$	1	$(3^4\ 4^2)$	130	(3^9)	460
$(5\ 6)$	1	$(3^7\ 4\ 5)$	4	$(3^4\ 4^4)$	3	$(3^3\ 4^2)$	50	(3^8)	386
$(3^3\ 4\ 6)$	1	$(3^6\ 4\ 5)$	9	$(3^3\ 4^4)$	4	$(3^2\ 4^2)$	16	(3^7)	209
$(3^2\ 4\ 6)$	1	$(3^5\ 4\ 5)$	22	$(3^2\ 4^4)$	3	$(3\ 4^2)$	6	(3^6)	90
$(3\ 4\ 6)$	1	$(3^4\ 4\ 5)$	22	$(3\ 4^4)$	2	(4^2)	2	(3^5)	32
$(4\ 6)$	2	$(3^3\ 4\ 5)$	18	(4^4)	1	$(3^{12}\ 4)$	1	(3^4)	14
$(3^6\ 6)$	4	$(3^2\ 4\ 5)$	8	$(3^9\ 4^3)$	1	$(3^{11}\ 4)$	10	(3^3)	5
$(3^5\ 6)$	3	$(3\ 4\ 5)$	5	$(3^8\ 4^3)$	1	$(3^{10}\ 4)$	54	(3^2)	2
$(3^4\ 6)$	6	$(4\ 5)$	2	$(3^7\ 4^3)$	8	$(3^9\ 4)$	242	(3)	1
$(3^3\ 6)$	5	$(3^{10}\ 5)$	1	$(3^6\ 4^3)$	23	$(3^8\ 4)$	515	$()$	1
$(3^2\ 6)$	4	$(3^9\ 5)$	2	$(3^5\ 4^3)$	48	$(3^7\ 4)$	599	total:	5250
$(3\ 6)$	2	$(3^8\ 5)$	16	$(3^4\ 4^3)$	57	$(3^6\ 4)$	399	time:	15 sec
(6)	1	$(3^7\ 5)$	33	$(3^3\ 4^3)$	36	$(3^5\ 4)$	180		
$(3^4\ 5^2)$	1	$(3^6\ 5)$	48	$(3^2\ 4^3)$	16	$(3^4\ 4)$	68		

TABLE VI. The Linear Spaces on 11 Points (Part I)

$v = 11$, line case	#	$v = 11$, line case	#	$v = 11$, line case	#	$v = 11$, line case	#	$v = 11$, line case	#
(11)	1	$(3^3\ 4\ 7)$	1	(7)	1	$(3\ 4^2\ 6)$	1	$(3^{10}\ 6)$	3
(10)	1	$(3^2\ 4\ 7)$	1	(6^2)	1	$(4^2\ 6)$	1	$(3^9\ 6)$	10
$(3\ 9)$	1	$(3\ 4\ 7)$	1	$(3^4\ 5\ 6)$	1	$(3^7\ 4\ 6)$	7	$(3^8\ 6)$	31
(9)	1	$(4\ 7)$	2	$(3^3\ 5\ 6)$	1	$(3^6\ 4\ 6)$	13	$(3^7\ 6)$	45
$(4\ 8)$	1	$(3^6\ 7)$	4	$(3^2\ 5\ 6)$	1	$(3^5\ 4\ 6)$	26	$(3^6\ 6)$	54
$(3^3\ 8)$	1	$(3^5\ 7)$	3	$(3\ 5\ 6)$	1	$(3^4\ 4\ 6)$	22	$(3^5\ 6)$	38
$(3^2\ 8)$	1	$(3^4\ 7)$	6	$(5\ 6)$	2	$(3^3\ 4\ 6)$	18	$(3^4\ 6)$	26
$(3\ 8)$	2	$(3^3\ 7)$	5	$(3^4\ 4^2\ 6)$	3	$(3^2\ 4\ 6)$	8	$(3^3\ 6)$	12
(8)	1	$(3^2\ 7)$	4	$(3^3\ 4^2\ 6)$	2	$(3\ 4\ 6)$	5	$(3^2\ 6)$	6
$(5\ 7)$	1	$(3\ 7)$	2	$(3^2\ 4^2\ 6)$	3	$(4\ 6)$	2	$(3\ 6)$	2

TABLE VII. The Linear Spaces on 11 Points (Part II)

$v = 11$, line case	#	$v = 11$, line case	#	$v = 11$, line case	#	$v = 11$, line case	#	time
(6)	1	$(3^{11}\ 4\ 5)$	3	$(3^{3}\ 4^{5})$	37	(4^{2})	2	
$(3^{6}\ 4\ 5^{2})$	1	$(3^{10}\ 4\ 5)$	43	$(3^{2}\ 4^{5})$	17	$(3^{15}\ 4)$	2	
$(3^{5}\ 4\ 5^{2})$	2	$(3^{9}\ 4\ 5)$	257	$(3\ 4^{5})$	7	$(3^{14}\ 4)$	35	
$(3^{4}\ 4\ 5^{2})$	5	$(3^{8}\ 4\ 5)$	792	(4^{5})	3	$(3^{13}\ 4)$	588	
$(3^{3}\ 4\ 5^{2})$	4	$(3^{7}\ 4\ 5)$	1244	$(3^{9}\ 4^{4})$	4	$(3^{12}\ 4)$	4169	
$(3^{2}\ 4\ 5^{2})$	3	$(3^{6}\ 4\ 5)$	1104	$(3^{8}\ 4^{4})$	30	$(3^{11}\ 4)$	13868	
$(3\ 4\ 5^{2})$	1	$(3^{5}\ 4\ 5)$	620	$(3^{7}\ 4^{4})$	147	$(3^{10}\ 4)$	24463	
$(4\ 5^{2})$	1	$(3^{4}\ 4\ 5)$	252	$(3^{6}\ 4^{4})$	430	$(3^{9}\ 4)$	24881	
$(3^{9}\ 5^{2})$	2	$(3^{3}\ 4\ 5)$	84	$(3^{5}\ 4^{4})$	595	$(3^{8}\ 4)$	15442	
$(3^{8}\ 5^{2})$	5	$(3^{2}\ 4\ 5)$	24	$(3^{4}\ 4^{4})$	486	$(3^{7}\ 4)$	6302	
$(3^{7}\ 5^{2})$	16	$(3\ 4\ 5)$	8	$(3^{3}\ 4^{4})$	210	$(3^{6}\ 4)$	1877	
$(3^{6}\ 5^{2})$	22	$(4\ 5)$	2	$(3^{2}\ 4^{4})$	60	$(3^{5}\ 4)$	466	
$(3^{5}\ 5^{2})$	29	$(3^{15}\ 5)$	1	$(3\ 4^{4})$	12	$(3^{4}\ 4)$	112	
$(3^{4}\ 5^{2})$	21	$(3^{14}\ 5)$	1	(4^{4})	3	$(3^{3}\ 4)$	28	
$(3^{3}\ 5^{2})$	12	$(3^{13}\ 5)$	8	$(3^{11}\ 4^{3})$	1	$(3^{2}\ 4)$	7	
$(3^{2}\ 5^{2})$	6	$(3^{12}\ 5)$	42	$(3^{10}\ 4^{3})$	44	$(3\ 4)$	2	
$(3\ 5^{2})$	4	$(3^{11}\ 5)$	238	$(3^{9}\ 4^{3})$	439	(4)	1	
(5^{2})	2	$(3^{10}\ 5)$	733	$(3^{8}\ 4^{3})$	1880	(3^{17})	2	
$(3^{3}\ 4^{4}\ 5)$	1	$(3^{9}\ 5)$	1381	$(3^{7}\ 4^{3})$	3983	(3^{16})	18	
$(3^{2}\ 4^{4}\ 5)$	1	$(3^{8}\ 5)$	1483	$(3^{6}\ 4^{3})$	4387	(3^{15})	237	2:59
$(3\ 4^{4}\ 5)$	1	$(3^{7}\ 5)$	1027	$(3^{5}\ 4^{3})$	2747	(3^{14})	1637	
$(4^{4}\ 5)$	1	$(3^{6}\ 5)$	502	$(3^{4}\ 4^{3})$	1033	(3^{13})	6198	
$(3^{6}\ 4^{3}\ 5)$	4	$(3^{5}\ 5)$	196	$(3^{3}\ 4^{3})$	272	(3^{12})	12735	
$(3^{5}\ 4^{3}\ 5)$	11	$(3^{4}\ 5)$	68	$(3^{2}\ 4^{3})$	57	(3^{11})	15418	
$(3^{4}\ 4^{3}\ 5)$	22	$(3^{3}\ 5)$	22	$(3\ 4^{3})$	12	(3^{10})	11623	
$(3^{3}\ 4^{3}\ 5)$	21	$(3^{2}\ 5)$	7	(4^{3})	4	(3^{9})	5754	
$(3^{2}\ 4^{3}\ 5)$	11	$(3\ 5)$	2	$(3^{13}\ 4^{2})$	3	(3^{8})	2033	
$(3\ 4^{3}\ 5)$	4	(5)	1	$(3^{12}\ 4^{2})$	54	(3^{7})	581	
$(4^{3}\ 5)$	1	$(3^{6}\ 4^{6})$	1	$(3^{11}\ 4^{2})$	732	(3^{6})	157	
$(3^{9}\ 4^{2}\ 5)$	6	$(3^{5}\ 4^{6})$	1	$(3^{10}\ 4^{2})$	4181	(3^{5})	44	
$(3^{8}\ 4^{2}\ 5)$	25	$(3^{4}\ 4^{6})$	2	$(3^{9}\ 4^{2})$	11301	(3^{4})	15	
$(3^{7}\ 4^{2}\ 5)$	120	$(3^{3}\ 4^{6})$	2	$(3^{8}\ 4^{2})$	16313	(3^{3})	5	
$(3^{6}\ 4^{2}\ 5)$	271	$(3^{2}\ 4^{6})$	2	$(3^{7}\ 4^{2})$	13282	(3^{2})	2	
$(3^{5}\ 4^{2}\ 5)$	327	$(3\ 4^{6})$	1	$(3^{6}\ 4^{2})$	6620	(3)	1	
$(3^{4}\ 4^{2}\ 5)$	229	(4^{6})	1	$(3^{5}\ 4^{2})$	2175	$()$	1	
$(3^{3}\ 4^{2}\ 5)$	100	$(3^{7}\ 4^{5})$	2	$(3^{4}\ 4^{2})$	559	total:	232929	
$(3^{2}\ 4^{2}\ 5)$	33	$(3^{6}\ 4^{5})$	7	$(3^{3}\ 4^{2})$	124	time:	17:02	
$(3\ 4^{2}\ 5)$	8	$(3^{5}\ 4^{5})$	23	$(3^{2}\ 4^{2})$	29			
$(4^{2}\ 5)$	4	$(3^{4}\ 4^{5})$	33	$(3\ 4^{2})$	7			

TABLE VIII. The Linear Spaces on 12 Points (Part I)

$v = 12$, line case	#	$v = 12$, line case	#	$v = 12$, line case	#	$v = 12$, line case	#
(12)	1	$(3^5\ 7)$	38	$(3^4\ 4^2\ 6)$	257	$(3^5\ 4^3\ 5^2)$	2
(11)	1	$(3^4\ 7)$	26	$(3^3\ 4^2\ 6)$	100	$(3^4\ 4^3\ 5^2)$	4
$(3\ 10)$	1	$(3^3\ 7)$	12	$(3^2\ 4^2\ 6)$	33	$(3^3\ 4^3\ 5^2)$	6
(10)	1	$(3^2\ 7)$	6	$(3\ 4^2\ 6)$	8	$(3^2\ 4^3\ 5^2)$	4
$(4\ 9)$	1	$(3\ 7)$	2	$(4^2\ 6)$	4	$(3\ 4^3\ 5^2)$	2
$(3^3\ 9)$	1	(7)	1	$(3^{12}\ 4\ 6)$	8	$(4^3\ 5^2)$	1
$(3^2\ 9)$	1	$(3^5\ 6^2)$	1	$(3^{11}\ 4\ 6)$	82	$(3^9\ 4^2\ 5^2)$	1
$(3\ 9)$	2	$(3^4\ 6^2)$	1	$(3^{10}\ 4\ 6)$	494	$(3^8\ 4^2\ 5^2)$	7
(9)	1	$(3^3\ 6^2)$	1	$(3^9\ 4\ 6)$	1391	$(3^7\ 4^2\ 5^2)$	43
$(5\ 8)$	1	$(3^2\ 6^2)$	1	$(3^8\ 4\ 6)$	2241	$(3^6\ 4^2\ 5^2)$	116
$(3^3\ 4\ 8)$	1	$(3\ 6^2)$	1	$(3^7\ 4\ 6)$	2137	$(3^5\ 4^2\ 5^2)$	175
$(3^2\ 4\ 8)$	1	(6^2)	2	$(3^6\ 4\ 6)$	1384	$(3^4\ 4^2\ 5^2)$	149
$(3\ 4\ 8)$	1	$(3^6\ 4\ 5\ 6)$	3	$(3^5\ 4\ 6)$	658	$(3^3\ 4^2\ 5^2)$	73
$(4\ 8)$	2	$(3^5\ 4\ 5\ 6)$	4	$(3^4\ 4\ 6)$	252	$(3^2\ 4^2\ 5^2)$	23
$(3^6\ 8)$	4	$(3^4\ 4\ 5\ 6)$	9	$(3^3\ 4\ 6)$	84	$(3\ 4^2\ 5^2)$	5
$(3^5\ 8)$	3	$(3^3\ 4\ 5\ 6)$	5	$(3^2\ 4\ 6)$	24	$(4^2\ 5^2)$	1
$(3^4\ 8)$	6	$(3^2\ 4\ 5\ 6)$	4	$(3\ 4\ 6)$	8	$(3^{12}\ 4\ 5^2)$	2
$(3^3\ 8)$	5	$(3\ 4\ 5\ 6)$	1	$(4\ 6)$	2	$(3^{11}\ 4\ 5^2)$	5
$(3^2\ 8)$	4	$(4\ 5\ 6)$	1	$(3^{15}\ 6)$	6	$(3^{10}\ 4\ 5^2)$	45
$(3\ 8)$	2	$(3^9\ 5\ 6)$	7	$(3^{14}\ 6)$	19	$(3^9\ 4\ 5^2)$	231
(8)	1	$(3^8\ 5\ 6)$	15	$(3^{13}\ 6)$	136	$(3^8\ 4\ 5^2)$	748
$(6\ 7)$	1	$(3^7\ 5\ 6)$	43	$(3^{12}\ 6)$	554	$(3^7\ 4\ 5^2)$	1354
$(3^4\ 5\ 7)$	1	$(3^6\ 5\ 6)$	47	$(3^{11}\ 6)$	1575	$(3^6\ 4\ 5^2)$	1443
$(3^3\ 5\ 7)$	1	$(3^5\ 5\ 6)$	53	$(3^{10}\ 6)$	2665	$(3^5\ 4\ 5^2)$	909
$(3^2\ 5\ 7)$	1	$(3^4\ 5\ 6)$	31	$(3^9\ 6)$	2922	$(3^4\ 4\ 5^2)$	390
$(3\ 5\ 7)$	1	$(3^3\ 5\ 6)$	18	$(3^8\ 6)$	2132	$(3^3\ 4\ 5^2)$	119
$(5\ 7)$	2	$(3^2\ 5\ 6)$	8	$(3^7\ 6)$	1178	$(3^2\ 4\ 5^2)$	33
$(3^4\ 4^2\ 7)$	3	$(3\ 5\ 6)$	5	$(3^6\ 6)$	523	$(3\ 4\ 5^2)$	8
$(3^3\ 4^2\ 7)$	2	$(5\ 6)$	2	$(3^5\ 6)$	195	$(4\ 5^2)$	4
$(3^2\ 4^2\ 7)$	3	$(3^3\ 4^4\ 6)$	2	$(3^4\ 6)$	68	$(3^{13}\ 5^2)$	5
$(3\ 4^2\ 7)$	1	$(3^2\ 4^4\ 6)$	2	$(3^3\ 6)$	22	$(3^{12}\ 5^2)$	30
$(4^2\ 7)$	1	$(3\ 4^4\ 6)$	1	$(3^2\ 6)$	7	$(3^{11}\ 5^2)$	235
$(3^7\ 4\ 7)$	9	$(4^4\ 6)$	1	$(3\ 6)$	2	$(3^{10}\ 5^2)$	935
$(3^6\ 4\ 7)$	15	$(3^6\ 4^3\ 6)$	12	(6)	1	$(3^9\ 5^2)$	2155
$(3^5\ 4\ 7)$	26	$(3^5\ 4^3\ 6)$	26	$(3^9\ 5^3)$	1	$(3^8\ 5^2)$	2809
$(3^4\ 4\ 7)$	22	$(3^4\ 4^3\ 6)$	37	$(3^8\ 5^3)$	1	$(3^7\ 5^2)$	2267
$(3^3\ 4\ 7)$	18	$(3^3\ 4^3\ 6)$	27	$(3^7\ 5^3)$	4	$(3^6\ 5^2)$	1206
$(3^2\ 4\ 7)$	8	$(3^2\ 4^3\ 6)$	11	$(3^6\ 5^3)$	7	$(3^5\ 5^2)$	484
$(3\ 4\ 7)$	5	$(3\ 4^3\ 6)$	4	$(3^5\ 5^3)$	9	$(3^4\ 5^2)$	160
$(4\ 7)$	2	$(4^3\ 6)$	1	$(3^4\ 5^3)$	8	$(3^3\ 5^2)$	50
$(3^{10}\ 7)$	7	$(3^9\ 4^2\ 6)$	32	$(3^3\ 5^3)$	5	$(3^2\ 5^2)$	16
$(3^9\ 7)$	16	$(3^8\ 4^2\ 6)$	143	$(3^2\ 5^3)$	2	$(3\ 5^2)$	6
$(3^8\ 7)$	38	$(3^7\ 4^2\ 6)$	395	$(3\ 5^3)$	1	(5^2)	2
$(3^7\ 7)$	49	$(3^6\ 4^2\ 6)$	551	(5^3)	1	$(3^6\ 4^5\ 5)$	7
$(3^6\ 7)$	54	$(3^5\ 4^2\ 6)$	458	$(3^6\ 4^3\ 5^2)$	1	$(3^5\ 4^5\ 5)$	13

TABLE IX. The Linear Spaces on 12 Points (Part II)

$v = 12$, line case	#	time	$v = 12$, line case	#	time	$v = 12$, line case	#	time
$(3^4\ 4^5\ 5)$	31		$(3^{12}\ 4\ 5)$	22709	9:53	$(3^2\ 4^7)$	28	
$(3^3\ 4^5\ 5)$	30		$(3^{11}\ 4\ 5)$	79359	13:48	$(3\ 4^7)$	9	
$(3^2\ 4^5\ 5)$	21		$(3^{10}\ 4\ 5)$	155771	10:49	(4^7)	2	
$(3\ 4^5\ 5)$	6		$(3^9\ 4\ 5)$	181006	5:11	$(3^9\ 4^6)$	1	
$(4^5\ 5)$	3		$(3^8\ 4\ 5)$	131002	2:38	$(3^8\ 4^6)$	11	
$(3^9\ 4^4\ 5)$	2		$(3^7\ 4\ 5)$	61555		$(3^7\ 4^6)$	150	
$(3^8\ 4^4\ 5)$	27		$(3^6\ 4\ 5)$	20194		$(3^6\ 4^6)$	618	
$(3^7\ 4^4\ 5)$	211		$(3^5\ 4\ 5)$	5025		$(3^5\ 4^6)$	1165	
$(3^6\ 4^4\ 5)$	645		$(3^4\ 4\ 5)$	1079		$(3^4\ 4^6)$	1195	
$(3^5\ 4^4\ 5)$	1067		$(3^3\ 4\ 5)$	215		$(3^3\ 4^6)$	653	
$(3^4\ 4^4\ 5)$	933		$(3^2\ 4\ 5)$	44		$(3^2\ 4^6)$	209	
$(3^3\ 4^4\ 5)$	493		$(3\ 4\ 5)$	9		$(3\ 4^6)$	36	
$(3^2\ 4^4\ 5)$	138		$(4\ 5)$	2		(4^6)	8	
$(3\ 4^4\ 5)$	29		$(3^{16}\ 5)$	50		$(3^{11}\ 4^5)$	1	
$(4^4\ 5)$	5		$(3^{15}\ 5)$	1288	5:26	$(3^{10}\ 4^5)$	32	
$(3^{12}\ 4^3\ 5)$	1		$(3^{14}\ 5)$	10698	13:20	$(3^9\ 4^5)$	547	2:35
$(3^{11}\ 4^3\ 5)$	4		$(3^{13}\ 5)$	44625	29:29	$(3^8\ 4^5)$	4107	3:17
$(3^{10}\ 4^3\ 5)$	132		$(3^{12}\ 5)$	103037	11:58	$(3^7\ 4^5)$	12913	2:30
$(3^9\ 4^3\ 5)$	1159		$(3^{11}\ 5)$	142742	6:24	$(3^6\ 4^5)$	20458	
$(3^8\ 4^3\ 5)$	5218		$(3^{10}\ 5)$	124400	4:01	$(3^5\ 4^5)$	17722	
$(3^7\ 4^3\ 5)$	11649		$(3^9\ 5)$	71443		$(3^4\ 4^5)$	8804	
$(3^6\ 4^3\ 5)$	14495		$(3^8\ 5)$	28469		$(3^3\ 4^5)$	2562	
$(3^5\ 4^3\ 5)$	10308		$(3^7\ 5)$	8549		$(3^2\ 4^5)$	452	
$(3^4\ 4^3\ 5)$	4427		$(3^6\ 5)$	2128		$(3\ 4^5)$	58	
$(3^3\ 4^3\ 5)$	1168		$(3^5\ 5)$	487		(4^5)	8	
$(3^2\ 4^3\ 5)$	218		$(3^4\ 5)$	112		$(3^{13}\ 4^4)$	2	
$(3\ 4^3\ 5)$	30		$(3^3\ 5)$	28		$(3^{12}\ 4^4)$	59	
$(4^3\ 5)$	5		$(3^2\ 5)$	7		$(3^{11}\ 4^4)$	1610	4:01
$(3^{13}\ 4^2\ 5)$	12		$(3\ 5)$	2		$(3^{10}\ 4^4)$	16007	10:28
$(3^{12}\ 4^2\ 5)$	245		(5)	1		$(3^9\ 4^4)$	70072	13:00
$(3^{11}\ 4^2\ 5)$	2946	4:48	$(3^4\ 4^9)$	1		$(3^8\ 4^4)$	152955	10:46
$(3^{10}\ 4^2\ 5)$	16450	10:25	$(3^3\ 4^9)$	1		$(3^7\ 4^4)$	183527	5:33
$(3^9\ 4^2\ 5)$	47845	8:39	$(3^2\ 4^9)$	1		$(3^6\ 4^4)$	128392	2:23
$(3^8\ 4^2\ 5)$	76609	5:38	$(3\ 4^9)$	1		$(3^5\ 4^4)$	54284	
$(3^7\ 4^2\ 5)$	72094	2:42	(4^9)	1		$(3^4\ 4^4)$	14340	
$(3^6\ 4^2\ 5)$	41330		$(3^5\ 4^8)$	1		$(3^3\ 4^4)$	2482	
$(3^5\ 4^2\ 5)$	15266		$(3^4\ 4^8)$	3		$(3^2\ 4^4)$	330	
$(3^4\ 4^2\ 5)$	3896		$(3^3\ 4^8)$	3		$(3\ 4^4)$	37	
$(3^3\ 4^2\ 5)$	784		$(3^2\ 4^8)$	3		(4^4)	6	
$(3^2\ 4^2\ 5)$	135		$(3\ 4^8)$	2		$(3^{16}\ 4^3)$	2	
$(3\ 4^2\ 5)$	24		(4^8)	1		$(3^{15}\ 4^3)$	4	
$(4^2\ 5)$	6		$(3^6\ 4^7)$	10		$(3^{14}\ 4^3)$	129	2:39
$(3^{15}\ 4\ 5)$	3		$(3^5\ 4^7)$	32		$(3^{13}\ 4^3)$	3205	28:57
$(3^{14}\ 4\ 5)$	228	4:12	$(3^4\ 4^7)$	52		$(3^{12}\ 4^3)$	38516	1:15:59
$(3^{13}\ 4\ 5)$	3299	4:33	$(3^3\ 4^7)$	51		$(3^{11}\ 4^3)$	207467	1:49:17

TABLE X. The Linear Spaces on 12 Points (Part III)

$v=12$, l.c.	#	time	$v=12$, l.c.	#	time	$v=12$, l.c.	#	time
$(3^{10}\,4^3)$	569190	1:32:28	$(3^4\,4^2)$	1298		(3^{20})	5	
$(3^9\,4^3)$	868161	42:03	$(3^3\,4^2)$	203		(3^{19})	511	4:02
$(3^8\,4^3)$	785607	17:53	$(3^2\,4^2)$	37		(3^{18})	9805	8:53
$(3^7\,4^3)$	440069	6:34	$(3\,4^2)$	7		(3^{17})	80304	9:58
$(3^6\,4^3)$	157932	2:00	(4^2)	2		(3^{16})	339704	20:50
$(3^5\,4^3)$	37936		$(3^{18}\,4)$	65		(3^{15})	828552	26:35
$(3^4\,4^3)$	6629		$(3^{17}\,4)$	2408	2:17	(3^{14})	1244997	23:20
$(3^3\,4^3)$	956		$(3^{16}\,4)$	37112	15:41	(3^{13})	1208959	18:28
$(3^2\,4^3)$	133		$(3^{15}\,4)$	263167	22:45	(3^{12})	786898	11:17
$(3\,4^3)$	22		$(3^{14}\,4)$	984173	45:14	(3^{11})	354149	6:25
(4^3)	5		$(3^{13}\,4)$	2107730	2:03:00	(3^{10})	113973	2:15
$(3^{17}\,4^2)$	3		$(3^{12}\,4)$	2752767	1:25:13	(3^9)	27611	
$(3^{16}\,4^2)$	115		$(3^{11}\,4)$	2295917	1:14:35	(3^8)	5519	
$(3^{15}\,4^2)$	3844	4:30	$(3^{10}\,4)$	1268667	26:59	(3^7)	1039	
$(3^{14}\,4^2)$	52198	19:37	$(3^9\,4)$	479434	7:14	(3^6)	213	
$(3^{13}\,4^2)$	329825	55:18	$(3^8\,4)$	129353	2:01	(3^5)	51	
$(3^{12}\,4^2)$	1075978	1:42:01	$(3^7\,4)$	26675		(3^4)	16	
$(3^{11}\,4^2)$	1981420	1:21:08	$(3^6\,4)$	4690		(3^3)	5	
$(3^{10}\,4^2)$	2193743	45:00	$(3^5\,4)$	790		(3^2)	2	
$(3^9\,4^2)$	1530287	23:12	$(3^4\,4)$	148		(3)	1	
$(3^8\,4^2)$	696583	33:41	$(3^3\,4)$	30		$()$	1	
$(3^7\,4^2)$	214717	9:46	$(3^2\,4)$	7		total: 28872973		
$(3^6\,4^2)$	47489	2:17	$(3\,4)$	2		time: 25:42:56		
$(3^5\,4^2)$	8259		(4)	1				

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