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THE CLASSIFICATION OF $(42, 6)_8$ ARCS

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Abstract. It is known that 42 is the largest size of a 6-arc in the Desarguesian projective plane of order 8. In this paper, we classify these $(42, 6)_8$ arcs. Equivalently, we classify the smallest 3-fold blocking sets in PG$(2, 8)$, which have size 31.

1. Introduction and Statement of Results

A k-arc in a projective plane of order q is a set of points A such that $|A \cap l| \leq k$ for all lines l and $|A \cap l| = k$ for some line l. An s-fold blocking set in a projective plane of order q is a set of points B such that $|B \cap l| \geq s$ for all lines l and $|A \cap l| = s$ for some line l. If $k + s = q + 1$, then k-arcs and s-fold blocking sets are equivalent objects. Namely, the sets A and B are complements of each other in the set $\mathcal{V}$ of points of the plane.

A k-arc of size n in PG$(2, q)$ will be referred to as $(n, k)_q$-arc. Arcs of large size (equivalently, small blocking sets) are of interest. A k-arc in a plane is largest if there is no k-arc whose size is larger. It is known from [1] that 42 is the size of a largest 6-arc in PG$(2, 8)$ (equivalently, that the smallest 3-fold blocking set in PG$(2, 8)$ has size 31). In this paper, we classify the $(42, 6)_8$-arcs up to isomorphism

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(equivalently, we classify the 3-fold blocking sets of size 31). Our main result is as follows (see Section 2 for notation regarding field elements):

Theorem 1. *Up to projective equivalence, there are five* $(42, 6)$ *arcs in* $\mathrm{PG}(2, 8)$, *corresponding to five* 3*-fold blocking sets of size* 31. *The blocking sets are as follows (the arcs are obtained by taking complements). Let* $\mathcal{C}$ *be a conic in* $\mathrm{PG}(2, 8)$ *and let* N *be the nucleus of* $\mathcal{C}$.

1. *(Arc I) Choose two points* P_1, P_2 *on* $\mathcal{C}$ *and let* $P_3 = N$. *The union of the triangle determined by* P_1, P_2, P_3 *with the conic forms a blocking set. The stabilizer has order* 42. *This arc is due to Mason* [7].
2. *(Arc II) Choose three points* P_1, P_2, P_3 *on* $\mathcal{C}$. *The union of the triangle determined by* P_1, P_2, P_3 *with the conic and the nucleus forms another blocking set. The stabilizer has order* 18.
3. *(Arc III) Consider a Fano plane* $\Pi = \mathrm{PG}(2, 2)$ *inside* $\mathrm{PG}(2, 8)$. *Choose a point* $P \in \Pi$. *Consider all points in* $\mathrm{PG}(2, 8)$ *on the four lines of* Π *not through* P, *together with* P. *This set is a blocking set with stabilizer of order* 72.
4. *(Arc IV) Consider again a Fano plane* $\Pi = \mathrm{PG}(2, 2)$ *inside* $\mathrm{PG}(2, 8)$. *Let* P_1, P_2, P_3 *be three points outside* Π *such that the triangle determined by* P_1, P_2, P_3 *is disjoint from* Π. *The union of the points of the triangle determined by* P_1, P_2, P_3 *and the seven points of* Π *forms a blocking set. The stabilizer of this blocking set has order* 63.
5. *(Arc V) Let* τ *be the projectivity defined by the matrix* $\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, *switching the* x *and the* y *coordinates. The blocking set is given as orbits under the group generated by* τ *of the following set of points:*
$$\{(0, \gamma, 1) \mid \gamma \in \mathbb{F}_8\}$$
$$\cup\{(1, 1, 1), (1, 1, 0), (1, 3, 1), (1, 4, 1), (4, 5, 1), (1, 7, 1), (4, 6, 1), (1, 2, 1)\}$$
It has an automorphism group of order 2, *namely the group generated by* τ.

The remainder of the paper is devoted to the proof of this theorem.

2. Additional Notation

We wish to introduce the following notation. Let $\mathcal{V}$ and $\mathcal{B}$ be the set of points and lines of $\mathrm{PG}(2, 8)$, respectively (of size 73 each). Let A be a 6-arc, and let $B = \mathcal{V} \setminus A$ be its complement. A line l is called i-line if $|A \cap l| = i$. The set of i-lines is denoted as $\mathcal{L}_i$. For $i \in \mathbb{Z}_{\geq 0}$, let $a_i = |\mathcal{L}_i|$. The vector $(a_0, a_1, \ldots, a_6)$ is the *type* (or *line-type*) of the arc. We introduce the exponential notation i^{a_i} as a shorthand for $a_i = i$. If P is a point, (P) denotes the pencil of lines through P, i.e., the set of lines containing P. For a set of lines $\mathcal{L}$, define
$$[\mathcal{L}] = \bigcup_{l \in \mathcal{L}} l$$
the set of points covered.

In the following, we will be working in the finite field $\mathbb{F}_8$. To simplify notation, we make the following conventions. The field is created using a root ω of $x^3 + x^2 + 1$ over $\mathbb{F}_2$. The elements of the field are polynomials $a_2\omega^2 + a_1\omega + a_0$ with $a_i \in \mathbb{F}_2$ for $i = 0, 1, 2$. For the sake of simplicity, we use the binary representation of the coefficient vector (a_2, a_1, a_0) as $4a_2 + 2a_1 + a_0$ to represent any such field element. In particular, the elements 0 and 1 in $\mathbb{F}_8$ are represented by the integers 0 and 1. The Frobenius automorphism ϕ is the mapping $x \mapsto x^2$. We write xyz or (x,y,z) for the projective point $\mathbf{P}(x,y,z) = \langle (x,y,z) \rangle$ and $abc^\perp$ for the line $\{xyz \mid ax+by+cz=0\}$. For simplicity, we use the same symbol ϕ to denote the collineation $xyz \mapsto x^\phi y^\phi z^\phi$ of $\mathrm{PG}(2,8)$. Also, if $A=(a_{ij})$ is a matrix, we denote by A^ϕ the matrix whose i,j entry is $a_{i,j}^\phi$. A semilinear map of $\mathrm{PG}(2,8)$ is a pair $f=(f_1,f_2)$ where f_1 is an element of $\mathrm{PGL}(3,8)$ and $f_2 \in \langle \phi \rangle$. If f_1 is represented by the 3×3 matrix A and $f_2 = \phi^i$ we write $f = A_i$ (i.e., we add i as a subscript to the matrix A). The action of f on a point xyz is given by

$$xyz \cdot f = (xyzA)^{\phi^i}.$$

Two semilinear maps $f = A_i$ and $g = B_j$ are composed according to the following rule:

$$A_i \cdot B_j = C_k \quad \text{where } C = A \cdot B^{\phi^{-i}}, \text{and} \qquad k = i+j.$$

3. Parameters and Tactical Decompositions

We have the following well-known result regarding possibilities for the line-type.

Lemma 1. *For an (n,s)-arc in $\mathrm{PG}(2,q)$, the following equations hold:*

$$\sum_{i=0}^{s} a_i = \theta_2, \quad \sum_{i=1}^{s} i a_i = n\theta_1, \quad \sum_{i=2}^{s} \binom{i}{2} a_i = \binom{n}{2},$$

where $\theta_j = (q^{j+1}-1)/(q-1)$.

Lemma 2. $a_1 = 0$.

Proof. Suppose that $a_1 > 0$. Then there exists a 1-line l, and a point P such that $A \cap l = \{P\}$. Counting the points of the arc in two ways yields:

$$42 = |A| = |A \cap l| + \sum_{m \in (P), m \neq l} (|A \cap m| - 1) \leq 1 + 5 \cdot 8 = 41,$$

which is a contradiction. Thus $a_1 = 0$. □

Lemma 3. *According to Lemma 1 and 2, there are exactly 111 possible line-types of $(42,6)_8$ arcs.*

In the following, we will refer to the 111 line-types as Case 1 through Case 111. Let c_i be the number of i-lines through a point.

Lemma 4. *The possible point-types $\mathbf{p}_i$ of points on the arc are*

$$\mathbf{p}_1 = 6^8\,2,\ \mathbf{p}_2 = 6^7\,5\,3,\ \mathbf{p}_3 = 6^7\,4^2,\ \mathbf{p}_4 = 6^6\,5^2\,4,\ \mathbf{p}_5 = 6^5\,5^4.$$

Proof. The point types $\mathbf{p} = (c_6, c_5, c_4, c_3, c_2, c_0)$ on the arc satisfy

$$\sum_{i=2}^{6} (i-1)c_i = 41, \quad \sum_{i=0}^{6} c_i = 9.$$

□

Lemma 5. *The possible point-types* $\mathbf{q}_i$ *of points off the arc are*

$$\begin{array}{llll}
\mathbf{q}_1 = 6^7\,0^2, & \mathbf{q}_{11} = 6^5\,3^4, & \mathbf{q}_{21} = 6^3\,5^4\,2^2, & \mathbf{q}_{31} = 6^2\,5^4\,4\,3^2,\\
\mathbf{q}_2 = 6^6\,4\,2\,0, & \mathbf{q}_{12} = 6^4\,5^3\,3\,0, & \mathbf{q}_{22} = 6^3\,5^3\,4\,3\,2, & \mathbf{q}_{32} = 6^2\,5^3\,4^3\,3,\\
\mathbf{q}_3 = 6^6\,3^2\,0, & \mathbf{q}_{13} = 6^4\,5^2\,4^2\,0, & \mathbf{q}_{23} = 6^3\,5^3\,3^3, & \mathbf{q}_{33} = 6^2\,5^2\,4^5,\\
\mathbf{q}_4 = 6^6\,2^3, & \mathbf{q}_{14} = 6^4\,5^2\,4\,2^2, & \mathbf{q}_{24} = 6^3\,5^2\,4^3\,2, & \mathbf{q}_{34} = 6\,5^6\,4\,2,\\
\mathbf{q}_5 = 6^5\,5^2\,2\,0, & \mathbf{q}_{15} = 6^4\,5^2\,3^2\,2, & \mathbf{q}_{25} = 6^3\,5^2\,4^2\,3^2, & \mathbf{q}_{35} = 6\,5^6\,3^2,\\
\mathbf{q}_6 = 6^5\,5\,4\,3\,0, & \mathbf{q}_{16} = 6^4\,5\,4^2\,3\,2, & \mathbf{q}_{26} = 6^3\,5\,4^4\,3, & \mathbf{q}_{36} = 6\,5^5\,4^2\,3,\\
\mathbf{q}_7 = 6^5\,5\,3\,2^2, & \mathbf{q}_{17} = 6^4\,5\,4\,3^3, & \mathbf{q}_{27} = 6^3\,4^6, & \mathbf{q}_{37} = 6\,5^4\,4^4,\\
\mathbf{q}_8 = 6^5\,4^3\,0, & \mathbf{q}_{18} = 6^4\,4^4\,2, & \mathbf{q}_{28} = 6^2\,5^6\,0, & \mathbf{q}_{38} = 5^8\,2,\\
\mathbf{q}_9 = 6^5\,4^2\,2^2, & \mathbf{q}_{19} = 6^4\,4^3\,3^2, & \mathbf{q}_{29} = 6^2\,5^5\,3\,2, & \mathbf{q}_{39} = 5^7\,4\,3,\\
\mathbf{q}_{10} = 6^5\,4\,3^2\,2, & \mathbf{q}_{20} = 6^3\,5^4\,4\,0, & \mathbf{q}_{30} = 6^2\,5^4\,4^2\,2, & \mathbf{q}_{40} = 5^6\,4^3.
\end{array}$$

Proof. The point types $\mathbf{q} = (c_6, c_5, c_4, c_3, c_2, c_0)$ off the arc satisfy

$$\sum_{i=0}^{6}(9-i-1)c_i = 30, \quad \sum_{i=0}^{6} c_i = 9.$$

□

Using Lemma 4 and Lemma 5, we observe for later use:

Corollary 1.
1. *2-lines do not intersect in A.*
2. *3-lines do not intersect in A.*
3. *No three zero-lines are concurrent.*
4. *No four 2-lines are concurrent.*
5. *The intersection point of two 0-lines is on exactly seven 6-lines and on no other lines (apart from the 0-lines).*
6. *Two 2-lines never intersect on a 0-line.*
7. *A 4-line and a 2-line intersect in B.*
8. *A 5-line and a 2-line intersect in B.*
9. *A 3-line intersects a 2-line in a point of B. Moreover, this point is on no other 0-line.*

Let $s_{i,j}$ be the value of c_j for the points of type $\mathbf{p}_i$ for $j = 0, \dots, 6$ and $i = 1, \dots, 5$ (and with $s_{i,1} = 0$). Let $t_{i,j}$ be the value of c_j for the points of type $\mathbf{q}_i$ for $j = 0, \dots, 6$ and $i = 1, \dots, 40$ (and with $t_{i,1} = 0$). Let x_i be the number of points of type $\mathbf{p}_i$ on the arc ($i = 1, \dots, 5$) and let y_i be the number of points of type $\mathbf{q}_i$ off the arc ($i = 1, \dots, 40$).

Lemma 6. *The x_i and y_i satisfy the following equations:*

$$\sum_{i=1}^{5} x_i \;=\; 42 \qquad (F_1), \qquad \sum_{i=1}^{40} y_i \;=\; 31 \qquad (F_2)$$

$$\sum_{i=1}^{5} x_i s_{i,j} \;=\; j a_j \qquad (F_{1,j}), \qquad \sum_{i=1}^{40} y_i t_{i,j} \;=\; (9-j) a_j \qquad (F_{2,j})$$

and

$$\begin{aligned} \sum_{i=1}^{5} x_i \binom{s_{i,j}}{2} + \sum_{i=1}^{40} y_i \binom{t_{i,j}}{2} &= \binom{a_j}{2} && (J_j) \\ \sum_{i=1}^{5} x_i s_{i,j_1} s_{i,j_2} + \sum_{i=1}^{40} y_i t_{i,j_1} t_{i,j_2} &= a_{j_1} a_{j_2} && (J_{j_1,j_2}) \end{aligned}$$

for $j, j_1, j_2 \in \{0, \dots, 6\}$ *with* $j_1 \neq j_2$.

Proof. The first two equations follow from the fact that each point on the arc (off the arc, resp.) has exactly on point type $\mathbf{p}_i$ ($\mathbf{q}_i$, resp.). The equations $(F_{1,j})$ and $(F_{2,j})$, resp., count the flags between points on the arc (off the arc, resp.) and j-lines. The equations (J_j) count pairs of intersecting j-lines, whereas (J_{j_1,j_2}) count ordered pairs of intersecting lines where the first line is a j_1-line and the second line is a j_2-line. □

We will refer to the equations in Lemma 6 as the *refinement equations.* They can be used to rule out cases of line types. If a line type has no refinement, no arc exists in that case.

One further set of necessary conditions is provided by the Johnson bound, which was generalized to tactical decompositions in [2]. For the Johnson bound itself, see [5],[3, Theorem 8],[6, Page 132].

Let $(\mathfrak{V}, \mathfrak{B})$ be a row-tactical decomposition of the linear space $\mathcal{S} = (\mathcal{V}, \mathcal{B})$. Let $\mathfrak{V} = (V_1, \dots, V_m)$ and $\mathfrak{B} = (B_1, \dots, B_n)$. Write $v_i = |V_i|$ ($i \in \mathbb{Z}_m$) and $b_j = |B_j|$ ($j \in \mathbb{Z}_n$). Let $r_{i,j}$ be the associated structure constants.

Lemma 7. *Let* $V_{i_1}, V_{i_2}, \dots, V_{i_s}$ *be a subset of classes of* $\mathfrak{V}$. *For* $j = 1, \dots, n$, *define* $w_j = \sum_{u=1}^{s} r_{i_u,j} v_{i_u}$ *and write* $w_j = f_j b_j + e_j$ *with* $0 \leq e_j < b_j$. *Assume that*

$$\sum_{j=1}^{n} \left\{ e_j \binom{f_j + 1}{2} + (b_j - e_j) \binom{f_j}{2} \right\} > \binom{\sum_{u=1}^{s} v_{i_u}}{2}.$$

Then the decomposition scheme is not realizable.

The cases of line-types for which at least one solution of the refinement equations of Lemma 6 exists are shown in Table 1. This list is 27 cases long. The column "# Sol" lists the number of solutions to the equations in Lemma 6. The column "# Ref" lists the number of refinements that survive the tests of Lemma 7. In particular, Cases 59 and 63 do not arise. This leaves 25 cases to consider.

Case	a_6	a_5	a_4	a_3	a_2	a_1	a_0	# Sol	# Ref	Comment
15	52	0	12	0	9	0	0	2	2	
37	49	8	3	8	4	0	1	23	15	
41	49	7	6	5	5	0	1	11	11	
44	51	0	15	0	6	0	1	2	2	
59	48	9	3	11	0	0	2	2	0	
63	48	8	6	8	1	0	2	21	0	
64	47	11	3	9	1	0	2	16	12	
68	48	7	9	5	2	0	2	32	18	
69	47	10	6	6	2	0	2	1351	1060	
70	46	13	3	7	2	0	2	13	9	
72	50	0	18	0	3	0	2	2	2	Arc V
75	47	9	9	3	3	0	2	197	196	
76	46	12	6	4	3	0	2	2139	1338	
77	45	15	3	5	3	0	2	2	2	
80	46	11	9	1	4	0	2	62	53	
81	45	14	6	2	4	0	2	112	80	
88	49	0	21	0	0	0	3	1	1	Arcs I & II
91	46	9	12	3	0	0	3	8	1	
92	45	12	9	4	0	0	3	214	32	
93	44	15	6	5	0	0	3	188	11	
94	43	18	3	6	0	0	3	4	2	
95	42	21	0	7	0	0	3	1	1	Arc IV
97	45	11	12	1	1	0	3	33	3	
98	44	14	9	2	1	0	3	447	12	
99	43	17	6	3	1	0	3	77	8	
102	43	16	9	0	2	0	3	66	11	
108	39	24	6	0	0	0	4	1	1	Arc III

TABLE 1. Possible Line-Types with refinement

For illustrative purposes, we display the two row-tactical decompositions in Case 72:

$$
\begin{array}{c|cccc}
\multicolumn{5}{c}{\text{Case.72.1}} \\
\rightarrow & 50 & 18 & 3 & 2 \\
\hline
6 & 8 & 0 & 1 & 0 \\
36 & 7 & 2 & 0 & 0 \\
\hline
1 & 7 & 0 & 0 & 2 \\
6 & 6 & 1 & 1 & 1 \\
1 & 6 & 0 & 3 & 0 \\
10 & 5 & 3 & 0 & 1 \\
12 & 4 & 4 & 1 & 0 \\
1 & 3 & 6 & 0 & 0
\end{array}
\qquad
\begin{array}{c|cccc}
\multicolumn{5}{c}{\text{Case.72.2}} \\
\rightarrow & 50 & 18 & 3 & 2 \\
\hline
6 & 8 & 0 & 1 & 0 \\
36 & 7 & 2 & 0 & 0 \\
\hline
1 & 7 & 0 & 0 & 2 \\
6 & 6 & 1 & 1 & 1 \\
10 & 5 & 3 & 0 & 1 \\
3 & 5 & 2 & 2 & 0 \\
9 & 4 & 4 & 1 & 0 \\
2 & 3 & 6 & 0 & 0
\end{array}
\tag{1}
$$

The remainder of the proof consists of examining the cases of Table 1. We will distinguish cases according to the number of 0-lines in the line-type (i.e., the value of a_0).

4. The Classification Algorithm

We wish to describe the classification algorithm that we will apply in Sections 6 and 7.

In order to decide whether a case with $1 \le a_0 \le 2$ exists, we wish to explore the intersection type of 0-lines with one other type of lines. This gives information that is useful for establishing a search.

More precisely, we choose w with $2 \le w < 6$ and look at the way in which lines from $\mathcal{L}_0$ intersect lines form $\mathcal{L}_w$. That is, we consider the set

$$S = \{l \cap m \mid l \in \mathcal{L}_0,\ m \in \mathcal{L}_w\} = [\mathcal{L}_0] \cap [\mathcal{L}_w].$$

The elements of the set S need to be counted with multiplicities. In order to facilitate this, we will introduce some notation on partitions.

For a set of lines $\mathcal{L}$ and for a point P, let $m_P(\mathcal{L})$ be the number of lines $l \in \mathcal{L}$ with $P \in l$. An i-point (with respect to $\mathcal{L}$) is a point P with $m_P(\mathcal{L}) = i$. A single (double, triple, etc.) point is an i-point with $i = 1$ ($i = 2, 3$ etc.). Also, for a set $X \subseteq \mathcal{V}$, let $M_i(X; \mathcal{L})$ be the set of i-points in X and let $m_i := m_i(X; \mathcal{L}) = |M_i(X; \mathcal{L})|$. Finally, let $\mu_{X;\mathcal{L}}$ be the (number-) partition $(1^{m_1}, 2^{m_2}, \ldots)$. For the sake of simplicity, we write $m_i(\mathcal{L})$ for $m_i(\mathcal{V}; \mathcal{L})$ and $\mu_{\mathcal{L}}$ for $\mu_{\mathcal{V};\mathcal{L}}$.

We write $\mu(i)$ for m_i in the partition $\mu = (1^{m_1}, \ldots)$. Also, for partitions $\mu = (1^{m_1}, 2^{m_2}, \ldots)$ and $\nu = (1^{n_1}, 2^{n_2}, \ldots)$, define a partition $\mu + \nu$ by putting

$$(\mu + \nu)(i) = \mu(i) + \nu(i) \quad \text{for all } i.$$

We define

$$B^* = B \setminus [\mathcal{L}_0]$$

the set of points off the arc that are not on 0-lines.

Example 1. Consider the two decompositions Case72.1 and Case72.2 from (1). Then $\mu_{[\mathcal{L}_0];\mathcal{L}_2} = 1^6$. Also, $\mu_{A;\mathcal{L}_2} = 1^6$ in both cases, while $\mu_{B^*;\mathcal{L}_2} = 1^{12}\,3$ in Case 72.1 and $\mu_{B^*;\mathcal{L}_2} = 1^9\,2^3$ in Case 72.2.

Lemma 8. *For $i \ne j$, $|\mu_{[\mathcal{L}_i];\mathcal{L}_j}| = a_i a_j$.*

Proof. Double count the set

$$\{(P, l, m) \mid P \in \mathcal{V},\ l \in \mathcal{L}_i,\ m \in \mathcal{L}_j \mid P = l \cap m\}.$$

□

We wish to describe the steps involved in the classification algorithm in some more detail. We start from the decompositions that are computed in Lemma 6 (and listed in Table 1). We choose an integer w with $2 \le w < 6$ and compute the possibilities for

1. $\mu_{[\mathcal{L}_0];\mathcal{L}_w}$ with $|\mu_{[\mathcal{L}_0];\mathcal{L}_w}| = a_0 a_w$.
2. $\mu_{A;\mathcal{L}_w}$
3. $\mu_{B^*;\mathcal{L}_w}$

It is important to note that, at this point, these numbers are just parameters. Namely, the sets $\mathcal{L}_0, \mathcal{L}_w$ and A and B have not yet been determined. The challenge is to find sets $\mathcal{L}_0, \mathcal{L}_w$ of lines and to partition $\mathcal{V}$ into A and B such that the parameters are satisfied. Observe that even if this can be done, the resulting set may not be an arc. Also, if it is an arc, it may or may not have the line type that we are considering. This is because we enforce only parameters related to two sorts of lines, namely $\mathcal{L}_0$ and $\mathcal{L}_w$, but we do not enforce the other parameters that are part of the tactical decomposition (this would be too much effort). So, it may happen that we end up with arcs with the wrong line type, in which case we just disregard those arcs.

Next we describe the search and classification algorithm. Initially, we let A and B be the empty set.

First, we select a_0 lines from $\mathcal{B}$, the lines of $\mathrm{PG}(2,8)$, to form the set $\mathcal{L}_0$. This is done up to equivalence of the group $\mathrm{P\Gamma L}(2,8)$, using an orbit algorithm. Let H be the stabilizer of these lines. We add $[\mathcal{L}_0]$ to the set B.

Because of Corollary 1, double points of $\mathcal{L}_0$ do not lie on w-lines for $2 \leq w < 6$. Therefore, $S \cap M_2(\mathcal{L}_0) = \emptyset$. Thus, in the second step, we choose $\|\mu_{[\mathcal{L}_0];\mathcal{L}_w}\|$ points from the set $[\mathcal{L}_0] \setminus M_2(\mathcal{L}_0)$ to form the set S. The possibilities for S are considered up to H-equivalence (again, using an orbit-algorithm). Let K be the stabilizer of the chosen set S in the group H.

In the third step, we choose a_w lines from $\mathcal{B} \setminus \mathcal{L}_0$ to serve as the set $\mathcal{L}_w$. We test if $\mathcal{L}_w$ and $\mathcal{L}_0$ intersect in the set S with the appropriate multiplicities. Also, we make sure that

$$\sum_{P \in l \cap S} m_P(\mathcal{L}_w) = a_w$$

for all $l \in \mathcal{L}_0$. Furthermore, we require that the intersection type of $\mathcal{L}_w$ on the set $\mathcal{V} \setminus S$ is $\mu_{\mathcal{V}\setminus S;\mathcal{L}_w} = \mu_{A;\mathcal{L}_w} + \mu_{B^*;\mathcal{L}_w}$. The possibilities for $\mathcal{L}_w$ are computed up to K-equivalence (using another instance of the orbit algorithm). Let L be the stabilizer of the lines $\mathcal{L}_w$ in the group K.

In step four, we consider the points in $T = [\mathcal{L}_w] \setminus S$. Consider $M_d(T, \mathcal{L}_w)$, the set of d-points in T. If for some $d > 0$ we find that $\mu_{A;\mathcal{L}_w}(d) > 0$ but $\mu_{B^*;\mathcal{L}_w}(d) = 0$, then all d-points in T are added to A. Conversely, if $\mu_{A;\mathcal{L}_w}(d) = 0$ but $\mu_{B^*;\mathcal{L}_w}(d) > 0$, then all d-points in T are added to B. There is no search in this step. Let D be the set of d in $1 \leq d \leq a_w$ for which $\mu_{A;\mathcal{L}_w}(d) > 0$ and $\mu_{B^*;\mathcal{L}_w}(d) > 0$.

In step five, we consider the remaining points in T. These are the points with multiplicity $d \in D$. From the third step, we know that $d_A + d_{B^*} = \mu_{\mathcal{V}\setminus S;\mathcal{L}_w}(d)$ for all $d \in D$. Thus, in step five, we choose d_A points from $M_d(T, \mathcal{L}_w)$ as candidates for addition to the set A. The remaining points of $M_d(T, \mathcal{L}_w)$ are added to the set B. This step is a backtrack search over all subsets of size d_A from $M_d(T, \mathcal{L}_w)$ and over all $d \in D$. One all choices have been made, we test whether $|m \cap B| = 9 - w$ for each line $m \in \mathcal{L}_w$. If it is not, we discard this possibility. Otherwise, we move on to the next step.

In step six, it remains to consider the points in the set

$$R = \mathcal{V} \setminus ([\mathcal{L}_0] \cup T).$$

The sizes $r_A = |R \cap A|$ and $r_B = |R \cap B|$ are known from the row-tactical decomposition. Thus, we perform one more backtrack search to choose r_B points out of R for addition to B.

If $a_0 = 2$, the described algorithm is not fast enough. Thus, in this case we modify step six as follows: Let P be the point of intersection of the two 0-lines. Let $l_1, \ldots, l_7$ be the lines in $(P) \setminus \mathcal{L}_0$. Let $R_i = R \cap l_i$ for $i = 1, \ldots, 7$. Let $f_i = 9 - |l_i \cap B'|$ so that $r_B = f_1 + \cdots + f_7$. If one f_i is negative, we discard the case. Otherwise, we proceed by choosing in all possible ways f_i points from the set R_i for $i = 1, \ldots, 7$ and adding the selected points to the set B'. This search is a backtrack search over $i = 1, \ldots, 7$.

In step seven, we consider the sets A and B that we obtained from the search. We test if A is a 6-arc (equivalently, if B is a 3-fold blocking set). If so, we compute its line type and check whether it is the same as the case we are currently considering.

5. AT LEAST THREE 0-LINES

We know from Lemma 3 that $a_0 \leq 4$. We first treat the case $a_0 = 4$.

Lemma 9. *(Case 108, Arc III) If $a_0 = 4$, there is up to projective equivalence exactly one $(42,6)_8$ arc. This arc has line type $0^4, 4^6, 5^{24}, 6^{39}$ and an automorphism group of order 72. This is Arc III.*

Proof. Let l_1, l_2, l_3, l_4 be the 0-lines. By Lemma 5, no three of the 0-lines are concurrent, so $\{l_1, l_2, l_3, l_4\}$ is a quadrilateral. Let $Q_{ij} = l_i \cap l_j$ for $1 \leq i < j \leq 4$ be the six intersection points. The three lines $\overline{Q_{14}, Q_{23}}$, $\overline{Q_{13}, Q_{24}}$, and $\overline{Q_{12}, Q_{34}}$ are concurrent in a point Q. The six points Q_{ij} together with Q form a Fano plane $\pi = \mathrm{PG}(2,2)$.

Since $[l_1, \ldots, l_4]$ contains 30 points of B, there is one additional point Q_0, in B, not on any of the lines $l_1, \ldots, l_4$. If $Q \neq Q_0$, then one of the three lines of π through Q must be a 7-line, a contradiction. Hence $Q = Q_0$. Note that the six lines through Q other than the lines of π are 4-lines. Since quadrilaterals are unique up to projective equivalence and since Q is uniquely determined by the 4 lines $l_1, \ldots, l_4$, this arc is unique. □

Now we consider the case $a_0 = 3$. Here we have the following results:

Lemma 10. *(Cases 88 and 95)*

1. *Up to projective equivalence, there are exactly two $(42,6)_8$-arcs with line type $0^3, 4^{21}, 6^{49}$. These are Arc I and Arc II.*
2. *Up to projective equivalence, there is exactly one $(42,6)_8$-arc with line type $0^3, 3^7, 5^{21}, 6^{42}$.*

Proof. Let l_1, l_2, l_3 be the 0-lines. Since the 0-lines are not concurrent, $P_1 := l_2 \cap l_3$, $P_2 := l_3 \cap l_1$, $P_3 := l_1 \cap l_2$ are three distinct points. Apart from the 24 points on $l_1 \cup l_2 \cup l_3$, there are seven points $Q_1, \ldots, Q_7$ in B^*. By Lemma 5, each of P_1, P_2, P_3 lies on seven 6-lines. If $P_h = l_i \cap l_j$ with h, i, j a permutation of $\{1,2,3\}$, these are the lines in the pencil other than l_i, l_j. Thus, $|l \cap B^*| = 1$ for each line $l \in (P_h)$, $l \neq l_i$, $l \neq l_j$ and $\{h, i, j\} = \{1,2,3\}$.

A computer search shows that there are 133 possibilities for the set B^*. Under the stabilizer of the triangle through P_1, P_2, P_3, which is a group of order 882, these

Orbit	Length	properties of B^*	Order of Stabilizer	Arc
1	49	all collinear	$882/49 = 18$	III
2	49	$(7,2)$-arc	$882/49 = 18$	I
3	21	$(7,2)$-arc	$882/21 = 42$	II
4	14	$\mathrm{PG}(2,2)$	$882/14 = 63$	IV

TABLE 2. The four orbits

133 possibilities fall into 4 orbits (cf. Table 2). In orbit 1, of length 49, all points of B^* are collinear. This leads to Arc III, as described in Lemma 9. Observe that we distinguished three of the four 0-lines. It is true that the fourth line can be mapped to any of the other 0-lines. Therefore, the stabilizer of order 18 is a subgroup of index 4 in the automorphism group of the arc, which therefore is a group of order 72.

In orbit 2, of length 49, the points of B^* form a $(7,2)$-arc. Thus $B^* \cup \{P_1, P_2, P_3\}$ is a hyperoval, and we get a $(42,6)_8$-arc with line type $0^3, 4^{21}, 6^{49}$. Let N be the nucleus of the hyperoval $B^* \cup \{P_1, P_2, P_3\}$. For this orbit, $N \in \{P_1, P_2, P_3\}$, and we obtain Arc I (the Mason [7] arc). Since we stabilize two points on a conic, the automorphism group has order $2(q-1)h = 42$.

In orbit 3, of length 21, we have $N \notin \{P_1, P_2, P_3\}$. Thus, we stabilize three points on a conic, and hence obtain an arc with an automorphism group of order $6h = 18$. This is Arc II.

In orbit 4, of length 14, the set B^* is a Fano plane embedded in $\mathrm{PG}(2,8)$. This yields Arc IV with an automorphism group of order $882/14 = 63$.

The uniqueness of Arcs I, II and IV follows from the fact that the stabilizer of the triangle is transitive on the arcs of these types. □

6. TWO 0-LINES

Lemma 11. *Up to equivalence, we may choose the two 0-lines to be $X = 0$ and $Y = 0$. The stabilizer is a group of order 18816. The point of intersection is $P = 001$.*

Proof. The group $\mathrm{P\Gamma L}(3,8)$ is doubly transitive on lines. Therefore, stabilizing two lines yields a subgroup of index $73 \cdot 72/2$. The order of $\mathrm{P\Gamma L}(3,8)$ is $73 \cdot 72 \cdot 64 \cdot 49 \cdot 3 = 49448448$, and hence the order of the stabilizer of two lines is 18816. □

The following results rely on computations in the field $\mathbb{F}_8$. Recall that we use the numbers 0 through 7 to denote the field elements.

Lemma 12. *In $\mathrm{PG}(2,8)$, there is, up to equivalence, only one way to choose 6 points on two lines, three on each line, not containing the point of intersection of the two lines. The stabilizer is a cyclic group of order 6. We may take the six points 100, 101, 201 and 010, 011, 021 lying on the lines $X = 0$, and $Y = 0$. In this case, the stabilizer is generated by the element h of order 6, where*

$$h = \begin{pmatrix} 0 & 2 & 2 \\ 2 & 0 & 2 \\ 0 & 0 & 4 \end{pmatrix}_2 . \tag{2}$$

Line	$\mu_{[\mathcal{L}_0];\mathcal{L}_4}$	$\mu_{A;\mathcal{L}_4}$	$\mu_{B^*;\mathcal{L}_4}$	r_A	r_B
1	1^6	1^{12}	$1^6\,3$	30	7
2	1^6	$1^8\,2^2$	$1^7\,2$	30	6
3	$1^4\,2$	$1^8\,2^2$	1^9	32	0
4	$1^3\,3$	1^{12}	1^9	30	5

TABLE 3. Parameters for Case 64

Line	S	$\|K\|$
1	$\{100, 101, 201, 010, 011, 021\}$	6
2	$\{100, 101, 201, 010, 011, 021\}$	6
3	$\{100, 101, 201, 010, 011\}$	6
4	$\{100, 101, 201, 010\}$	21

Line	$\mathcal{L}_4$	$\mu_{\mathcal{V}\setminus S;\mathcal{L}_4}$	$\|L\|$
1	$\{001^\perp, 111^\perp, 661^\perp\}$	$1^{18}\,3$	6
2	$\{001^\perp, 161^\perp, 611^\perp\}$	$1^{15}\,2^3$	2
3	$\{001^\perp, 101^\perp, 661^\perp\}$	$1^{17}\,2^2$	1
4	$\{001^\perp, 101^\perp, 601^\perp\}$	1^{21}	21

TABLE 4. Results from the Search in Case 64

Proof. Recall from Lemma 11 that $\mathrm{PGL}(2,q)$ is two-transitive on lines and that the stabilizer of $X = 0$ and $Y = 0$ is a group H of order 18816. A direct verification shows that h maps 100 to 011 to 102 to 010 to 021 to 100, i.e. h acts on the given points as a 6-cycle, and therefore stabilizes the set. There are $\binom{8}{3}^2$ ways to choose sets of six points on $X = 0$ and $Y = 0$ satisfying the conditions. Since $18816/\binom{8}{3}^2 = 6$, the group H must be transitive on the sets of sixes satisfying the condition. Thus the full stabilizer is the cyclic group of order 6 generated by h. $\square$

Lemma 13. *The group* $\mathrm{PGL}(2,8)$ *is* 4*-homogeneous on the* 9 *points of* $\mathrm{PG}(1,8)$.

Proof. We have to show that $\mathrm{PGL}(2,8)$ is transitive on subsets of size 4. The elements of $\mathrm{PG}(1,8)$ are $\infty = \langle 1,0\rangle$ and $t = \langle t,1\rangle$ for $t \in \mathbb{F}_8$. The stabilizer of $\{\infty, 0, 1\}$ is a symmetric group H on three elements, generated by the mappings $\sigma : t \mapsto \frac{1}{t+1}$ (of order 3) and $\tau : t \mapsto \frac{1}{t}$ (of order two). A calculation shows that H is transitive on the remaining points, since $2 \mapsto 4 \mapsto 7$ and $6 \mapsto 5 \mapsto 3$ under σ and $2 \mapsto 6$ under τ. $\square$

Lemma 14. *Case 64 does not exist.*

Proof. The line type in Case 64 is $0^2, 2, 3^9, 4^3, 5^{11}, 6^{47}$. The 12 tactical decompositions from Table 1 lead to three possibilities for $\mu_{[\mathcal{L}_0];\mathcal{L}_4}$ and four possibilities for $\mu_{A;\mathcal{L}_4}$ and $\mu_{B^*;\mathcal{L}_4}$ (cf. Table 3).

We facilitate a computer search for each of the Lines 1 through 4 (cf. Table 4). By Lemma 11, we may assume $\mathcal{L}_0 = \{100^\perp, 010^\perp\}$ with a stabilizer H of order 18816.

In Lines 1 and 2, we have $\mu_{[\mathcal{L}_0];\mathcal{L}_4} = 1^6$. From Lemma 12 we may assume that $S = \{100, 101, 201, 010, 011, 021\}$ with stabilizer $K = H_S = \langle h\rangle \simeq C_6$ with

h as in (2). The three lines $001^\perp, 111^\perp, 661^\perp$ are concurrent in 110, and hence satisfy the intersection condition $\mu_{A;\mathcal{L}_4} + \mu_{B^*;\mathcal{L}_4}$. The stabilizer of the lines is $L = K_{\mathcal{L}_4} = K = H$. A search for arcs results in arcs with line types different from Case 64. The three lines $001^\perp, 161^\perp, 611^\perp$ intersect in $610, 210, 551$ and hence satisfy the intersection condition $\mu_{A;\mathcal{L}_4} + \mu_{B^*;\mathcal{L}_4}$. The stabilizer of the lines is $L = \langle h^3 \rangle \simeq C_2$, with

$$h^3 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}_0 .$$

A search for arcs results in arcs with line types different from Case 64.

In Line 3, the only possibility for the set S is $100, 101, 201$ on $Y = 0$ and $010, 011$ on $X = 0$. The stabilizer is cyclic of order 6, generated by

$$\begin{pmatrix} 1 & 0 & 5 \\ 0 & 5 & 5 \\ 0 & 0 & 5 \end{pmatrix}_1 .$$

The only possibility for lines is $001^\perp, 101^\perp, 611^\perp$ with a trivial stabilizer. A search for arcs results in arcs with line types different from Case 64.

In Line 4, the only possibility for the set S is $100, 101, 201$ on $Y = 0$ and 010 on $X = 0$. The stabilizer K is the nonabelian group of order 21 generated by

$$h_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 1 \end{pmatrix}_0 , \quad h_2 = \begin{pmatrix} 3 & 0 & 2 \\ 0 & 5 & 0 \\ 0 & 0 & 2 \end{pmatrix}_1 ,$$

with relations $h_1^7 = h_2^3 = 1$, $h_1^{h_2} = h_1^2$. The only possibility for lines is $001^\perp, 101^\perp, 601^\perp$ with stabilizer $L = K$. A computer search shows that there are no arcs in this case either. Thus Case 64 does not arise. □

Lemma 15. *Cases 68, 69 and 70 do not exist.*

Proof. The line types in Cases 68, 69 and 70 all have $a_2 = 2$. In all cases, the tactical decompositions lead to $\mu_{[\mathcal{L}_0];\mathcal{L}_2} = 1^4$ and $\mu_{A;\mathcal{L}_2} = 1^4$ and $\mu_{B^*;\mathcal{L}_2} = 1^8, 2$ with $r_A = 38$ and $r_B = 5$. By Lemma 11, we may assume $\mathcal{L}_0 = \{100^\perp, 010^\perp\}$ with a stabilizer H of order 18816. Up to equivalence, there is only one possibility for choosing the set S with two points on each 0-line, namely $100, 101$ on $Y = 0$ and $010, 011$ on $X = 0$. The stabilizer of these 4 points is the group H of order 24. This group is the 2-point set-stabilizer of order 8 in the group of the Fano plane, extended by the group of field automorphisms of order 3. Up to H-equivalence, there is only one possibility for choosing the 2-lines, namely $001^\perp$ and $111^\perp$. The stabilizer of these lines is a group K or order 12. A computer search shows that there are no arcs in these cases. □

Lemma 16. *Cases 75, 76 and 77 do not exist. Case 72 gives rise to a unique example, Arc V, invariant under the group $\langle h^3 \rangle \simeq C_2$ where h is as in Lemma 12.*

Proof. Cases 72, 75, 76 and 77 all have three 2-lines. We consider the possibilities for intersections of the lines in $\mathcal{L}_2$ with the two 0-lines. The tactical decompositions from Table 1 are examined by computer. The number of row-tactical decompositions in Cases 72 (75, 76 and 77 resp.) is 2 (196, 1338 and 2, resp.). (The two row-tactical decompositions in Case 72 are displayed in (1)). In all cases, the tactical decompositions lead to $\mu_{[\mathcal{L}_0];\mathcal{L}_2} = 1^6$, i.e., the 2-lines meet the 0-lines transversally. The three lines in $\mathcal{L}_2$ either form a triangle or are concurrent. In fact, the row-tactical decompositions from Table 1 show that we have the two possibilities $\mu_{A;\mathcal{L}_2} = 1^6$, $\mu_{B^*;\mathcal{L}_2} = 1^9, 2^3$, $r_A = 36$, $r_B = 2$ and $\mu_{A;\mathcal{L}_2} = 1^6$, $\mu_{B^*;\mathcal{L}_2} = 1^{12}, 3$, $r_A = 36$, $r_B = 1$. In the first case, the lines of $\mathcal{L}_2$ form a triangle in B. In the second case, the lines of $\mathcal{L}_2$ are concurrent in a point of B. The second case does not arise in Case 77.

From Lemma 12, we may assume that $S = \{100, 101, 201, 010, 011, 021\}$ with a stabilizer $H = \langle h \rangle \simeq C_6$, with h as in the lemma.

For the first possibility, we have two cases for the set of three lines $\mathcal{L}_2$. Either we have lines in the orbit of $011^\perp, 101^\perp, 661^\perp$ with a stabilizer $C_2 = \langle h^3 \rangle$ or in the orbit $011^\perp, 161^\perp, 601^\perp$ with a stabilizer $C_3 = \langle h^2 \rangle$. If the stabilizer is C_2, the three lines intersect in $111, 131, 311$, which are points in B. Computer search shows that this leads to Arc V in Case 72, and that this arc is unique. It also shows that the case where the the set $\mathcal{L}_2$ is stabilized by C_2 does not lead to an arc.

For the second possibility, there is no choice. Up to equivalence, the three 2-lines must be $001^\perp, 101^\perp, 661^\perp$ with a stabilizer $C_6 = \langle h \rangle$. Computer search shows that there is no arc in this case. □

It remains to consider Cases 80 and 81. To this end, we first consider basic results in PG$(2, 8)$.

Lemma 17. [4, Lemma 8.9] *In PG$(2, 8)$, if two hyperovals have at least 6 points in common, they coincide.*

Lemma 18. [4, Table 14.11] *For $7 \le k \le 9$, any $(k, 2)$-arc in PG$(2, 8)$ is contained in a unique $(10, 2)$-arc (or hyperoval).*

Lemma 19. *For any (8,3)-arc $\mathcal{L}$ of lines (i.e. an arc in the dual plane) in PG(2,8), $m_3(\mathcal{L}) \neq 1$. For any (9,3)-arc $\mathcal{L}$ of lines in PG(2,8), $m_3(\mathcal{L}) \neq 1, 2$.*

Proof. Assume that P is the unique triple point of $\mathcal{L} = \{l_1, \dots, l_8\}$ on l_1, l_2, l_3. Then $\mathcal{L} \setminus \{l_1\}$ and $\mathcal{L} \setminus \{l_2\}$ are $(7, 2)$-arcs of lines. By Lemma 18, there exist line-hyperovals H_1 and H_2 containing these line-arcs, respectively. Since H_1 and H_2 share the 6 lines $\mathcal{L} \setminus \{l_1, l_2\}$, Lemma 17 implies $H_1 = H_2 =: H$ and $\mathcal{L} \subseteq H$. Thus $\mathcal{L}$ is an (8,2)-arc of lines, too, and there cannot be a triple point. The second assertion follows from the first. □

Lemma 20. *Let $a_0 = 2$ and $a_2 = 4$. Then B is covered by the set of 0-lines and 2-lines. Moreover, no three lines of $\mathcal{L}_0 \cup \mathcal{L}_2$ are concurrent, i.e., the set $\mathcal{L}_0 \cup \mathcal{L}_2$ forms a $(6, 2)$-arc of lines.*

Proof. Let z_1 and z_2 be the two 0-lines, and let $l_1, \dots, l_4$ be the four 2-lines. Observe that z_1 and z_2 cover 17 points of B, so that $B^* := B \setminus (z_1 \cup z_2)$ is a set of 14 points.

Using Corollary 1, we deduce

$$\begin{aligned} 14 = |B^*| \geq & \left|\cup_{i=1}^4 (B^* \cap l_i)\right| \\ &= \sum_{i=1}^4 |B^* \cap l_i| - \sum_{1\leq i<j\leq 4} |B^* \cap l_i \cap l_j| + \sum_{1\leq i<j<k\leq 4} |B^* \cap l_i \cap l_j \cap l_k| \\ &= 4(9-2-2) - \binom{4}{2} + \sum_{1\leq i<j<k\leq 4} |B^* \cap l_i \cap l_j \cap l_k| \\ &= 14 + \sum_{1\leq i<j<k\leq 4} |B^* \cap l_i \cap l_j \cap l_k| \,. \end{aligned}$$

Thus $B^* = \cup_{i=1}^4 (B^* \cap l_i)$ and therefore $B^* \subseteq \cup_{i=1}^4 l_i$. Also,

$$\sum_{1\leq i<j<k\leq 4} |B^* \cap l_i \cap l_j \cap l_k| = 0,$$

which means that three 2-lines are never concurrent. To finish the proof, we use Corollary 1 once more. Two 0-lines intersect in a point that is on no 2-line. Also, two 2-lines never intersect on a 0-line. □

Lemma 21. *Assume $|\mathcal{L}_0| = 2$ and $|\mathcal{L}_2| = 4$. Let d be a 3-line. Then $\mathcal{L}_0 \cup \mathcal{L}_2 \cup \{d\}$ is a $(7,2)$-arc of lines.*

Proof. We know that $\mathcal{L}_0 \cup \mathcal{L}_2$ is a line arc, so we need to consider the intersections of d with these six lines. From Corollary 1 it follows that the 3-line is never concurrent with one 0-line and one 2-line. Assume that Q is the point of intersection of d with two 2-lines. This means that Q is a point of type $\mathbf{q}_7$ as in Lemma 5. Since $|d \cap B| = 6$, and since two 2-lines intersect d in Q, the other four 0-lines and 2-lines cannot cover all of the remaining five point of $d \cap B \setminus \{Q\}$. Thus we find a point of B that is not on either a 0-line or a 2-line, in contradiction to Lemma 20. □

Lemma 22. *Cases 80 and 81 do not arise.*

Proof. Let $K = \mathcal{L}_0 \cup \mathcal{L}_2 \cup \{d\}$ where d is a 3-line. By Lemma 21, K is a (7,2)-arc of lines. Then K has $m_1(K) = 7 \cdot 3 = 21$ 1-points, of which $3 + 2 \cdot 4 = 11$ are in A by Corollary 1, and these 21 points are disjointly covered by the lines d_1, d_2, d_3 by Lemma 18. By Lemma 20, all points in $d_i \setminus K$ are in A. Hence, among $\{d_1, d_2, d_3\}$, one is a 5-line, say d_1, and the other two are 6-lines. By Corollary 1, 5-line d_1 intersect the four 2-lines and the two 0-lines in 6 distinct points in B, which is a contradiction. □

7. At Most One 0-Line

Lemma 23. *Cases 37, 41 and 44 do not exist.*

Proof. We consider the intersections of the w-lines with the unique 0-line for a chosen value of w. The possible parameters are listed in Table 5 In Cases 37 and 41, we need to choose the set S of size 5. In Case 44, we need to choose the set S of size 6. By Lemma 13, this can be done in one way only (considering the complement of the set we wish to choose). The stabilizer of the set S is a group H of order 5376

Case	Line-Case	w	$\mu_{[\mathcal{L}_0];\mathcal{L}_w}$	$\mu_{A;\mathcal{L}_w}$	$\mu_{B^*;\mathcal{L}_w}$	r_A	r_B	$\_a\_$
1	37	3	$1^2\,2^3$	1^{24}	$1^7\,2^{10}\,3^3\,4$	18	1	240
2	37	3	$1^2\,2^3$	1^{24}	$1^4\,2^{13}\,3^2\,4$	18	2	402
3	37	3	$1^2\,2^3$	1^{24}	$1\,2^{16}\,3\,4$	18	3	149
4	41	2	1^5	1^{10}	$1^{13}\,2^7\,3$	32	1	4
5	41	2	1^5	1^{10}	$1^{10}\,2^{10}$	32	2	5
6	44	2	1^6	1^{12}	$1^6\,2^{15}$	30	1	3
7	44	2	1^6	1^{12}	$1^9\,2^{12}\,3$	30	0	8

TABLE 5. Parameters for Cases 37, 41, 44

in Cases 37 and 41 and of order 8064 in Case 44. There are three possibilities for $\mu_{A;\mathcal{L}_w}$ and $\mu_{B^*;\mathcal{L}_w}$ in Case 37 (and two possibilities in each of Case 41 and Case 44). The number of orbits of H on sets of 8 lines satisfying the intersection conditions $\mu_{A;\mathcal{L}_w} + \mu_{B^*;\mathcal{L}_w}$ are listed in column $\_a\_$. A computer search shows that none of the Cases 37, 41 or 44 leads to an arc. Thus these cases do not arise. □

It remains to treat Case 15 with no 0-line.

Lemma 24. *Case 15 does not arise.*

Proof. In Case 15, the nonzero a_i are $a_2 = 9$, $a_4 = 12$, and $a_6 = 52$. Calculations show that points in A are of type $6^8\,2$ or $6^7\,4^2$. Also, the possible types of points in B are $6^6\,2^3$, $6^5\,4^2\,2^2$, $6^4\,4^4\,2$ and $6^3\,4^6$.

We observe that any point in B is on an even number of 4-lines. This implies the following:

Claim 1: The points in A on a 4-line are all of type $6^7\,4^2$, while there are four points in B of type $6^5\,4^2\,2^2$ and one point of type $6^4\,4^4\,2$. Moreover, no point in B has type $6^3\,4^6$.

Claim 2: The number of the points of type $6^4\,4^4\,2$ in $\mathrm{PG}(2,8)$ is exactly three. These three points are collinear in a line l, and l is not a 4-line.

It follows from Claim 1 that there are three points P_1, P_2, P_3 of type $6^4\,4^4\,2$ that each lie on four 4-lines. Let l be the line containing P_1 and P_2. We must show that P_3 is also on l. Assume the contrary. Since l contains two points of type $6^4\,4^4\,2$, by Claim 1, l is not a 4-line. Thus $l \in \mathcal{L}_2$ or $l \in \mathcal{L}_6$. Let Q be the point of intersection of l with any 4-line through P_3. Then Q is on only one 4-line, since the other 4-lines pass through one of the P_i and hence not through Q. But this is a contradiction since every point on a 4-line is on another 4-line. Thus l contains P_1, P_2 and P_3.

Claim 3: $m_1(\mathcal{L}_2) = 21$, $m_2(\mathcal{L}_2) = 24$, $m_3(\mathcal{L}_2) = 4$, and $m_i(\mathcal{L}_2) = 0$ for $i \geq 4$.

Solving the equations

$$\sum_{i=1}^{3} m_i(\mathcal{L}_2) = 49, \quad \sum_{i=1}^{3} i m_i(\mathcal{L}_2) = 81, \quad \sum_{i=2}^{3} \binom{i}{2} m_i(\mathcal{L}_2) = \binom{9}{2}$$

leads to the desired statement.

Claim 4: The four triple points of $\mathcal{L}_2$ are collinear.

If the line l in Claim 2 is a 2-line, three points of type $6^4\,4^4\,2$ are on $l \cap B$. Thus the other points of l do not lie on 4-lines and hence $l \cap B$ contains four points of type $6^6\,2^3$ which are the four triple points. Now it remains to prove that l can not be a 6-line. Suppose, on the contrary, that it is a 6-line. Take a point on l of type $6^4\,4^4\,2$. This point is on a 2-line l' containing exactly one point of type $6^4\,4^4\,2$. Counting the number of 2-lines through each point in $l' \cap B$ we can show that l' contains exactly 2 triple points. Then $(\mathcal{L}_2 \setminus \{l'\}) \cup \{l\}$ is a (9,3)-arc of lines with exactly two triple point, which contradicts Lemma 19.

Since $K := \mathcal{L}_2 \setminus \{l\}$ is a (8,2)-arc of lines, K has $2 \cdot 8 = 16$ 1-points, which are in A by Corollary 1. By Lemma 18, there exist two lines d_1, d_2 such that $(\mathcal{L}_2 \setminus \{l\}) \cup \{d_1, d_2\}$ is a (10,2)-arc of lines. Then $d_1 \cup d_2$ should contain those 16 points in A, which contradicts the fact that A is (42,6)-arc.

□

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