Dr. A. Betten Fall 2008

MATH 501 Introduction to Combinatorial Theory

Assignment # 1

Problem # 1
Prove

$$\sum_{i=0}^{m} \binom{r}{i}\binom{s}{m-i} = \binom{r+s}{m}.$$

Hint: $(1+x)^r(1+x)^s = (1+x)^{r+s}$.

Problem # 2
A flag is to be designed with 13 horizontal stripes colored red, white, or blue, subject to the condition that no stripe be of the same color as the one above it. In how many ways can this be done?

Problem # 3
How many positive integers less than 10^n (in decimals) have their digits in nondecreasing order?

Problem # 4
What is the maximum length of a binary string (of 0's and 1's) such that no two substrings of length three are the same? Bonus: Give an example of such a string.

Problem # 5
Calculate the number of ways to form words (meaningful or not) from the letters of the following words:

a) WOLLONGONG

b) WAGGAWAGGA

c) MISSISSIPPI

d) BARACKOBAMA

e) JOHNMCCAIN

Two words are considered equal when they are spelled in the same way.

Problem # 6
Find a word (meaningful or not) that has exactly 277200 rearrangements.

Problem # 7
A box contains 20 cell phones, of which 4 are Nokia, 7 are Motorola and 9 are Samsung. What is the smallest number of cell phones which must be chosen (blindfolded) so that the selection is guaranteed to contain $r = 4, 5, 6, 7, 8, 9$ phones of the same make?

Problem # 8
Find the total number of positive integers with distinct digits.

Problem # 9
If a set X has $2n+1$ elements, find the number of subsets of X with at most n elements.

Problem # 10
A rumor is spread randomly among a group of 10 people by successively having one person call someone who calls someone, and so on. A person can pass the rumour on to anyone except the individual who just called.
(a) How many different paths can a rumor travel through the group in three calls? In n calls?
(b) What is the probability that if A starts the rumor, A receives the third call?
(c) What is the probability that if A does not start the rumor, A receives the third call?

Dr. A. Betten Fall 2008

MATH 501 Introduction to Combinatorial Theory

Assignment # 2

Problem # 11
Find the number of positive integers with n digits in which no two adjacent digits are the same. How many of these are even?

Problem # 12
The zebra problem: There are five houses in a row, each of a different color, inhabitated by women of different nationalities. The owner of each house owns a different pet, serves different drinks, and smokes different cigarettes from the other owners. The following facts are also known:
The Englishwomen lives in the red house.
The Spaniard owns a dog.
Coffee is drunk in the green house.
The Ukrainian drinks tea.
The green house is immediately to the right of the ivory house.
The Oldgold smoker owns the snail.
Kools are smoked in the yellow house.
Milk is drunk in the middle house.
The Norwegian lives in the first house on the left.
The Chersterfield smoker lives next to the fox owner.
The yellow house is next to the horse owner.
The Lucky Strike smoker drinks orange juice.
The Japanese smokes Parliament.
The Norwegian lives next to the blue house.

The question: Who drinks water and who owns the zebra?

Problem # 13
(a) How many different five-digit numbers are there (leading zeros, e.g., 00174, not allowed)?
(b) How many even five-digit numbers are there?
(c) How many five-digit numbers are there with exactly one 3?
(d) How many five-digit numbers are there that are the same when the order of the digits in inverted (e.g., 15251)?

Let f_n be the n-th Fibonacci number (i.e., $f_1 = f_2 = 1$ and $f_n = f_{n-1} + f_{n-2}$ for $n \geq 3$.)

Problem # 14
Prove that $f_1^2 + f_2^2 + \cdots + f_n^2 = f_n f_{n+1}$ whenever n is a positive integer.

Problem # 15
Prove that $f_1 + f_3 + \cdots + f_{2n-1} = f_{2n}$ whenever n is a positive integer.

Problem # 16
Show that $f_{n+1}f_{n-1} - f_n^2 = (-1)^n$ whenever n is a positive integer.

Problem # 17
The **McCarthy 91** function is defined using the rule

$$M(n) = \begin{cases} n - 10 & \text{if } n > 100 \\ M(M(n+11)) & \text{if } n \leq 100 \end{cases}$$

for all positive integers. By successively using the defining rule for $M(n)$, find
a) $M(102)$ b) $M(101)$ c) $M(99)$ d) $M(97)$ e) $M(87)$ f) $M(76)$.

Bonus Problem

Problem # 18
Show that the function $M(n)$ from the previous exercise if a well-defined function from the set of positive integers to the set of positive integers. *Hint:* Prove that $M(n) = 91$ for all positive integers n with $n \leq 101$.)

Dr. A. Betten Fall 2008

MATH 501 Introduction to Combinatorial Theory

Assignment # 3

Problem # 19

You have taken out a mortgage of $\$120,000$ at 7% fixed interest amortized over 30 years with fixed monthly payments.

a) How long (in months) will it take you to pay back the first $\$10,000$?

b) How long (in months) will it take you to pay back the last $\$10,000$?

c) After 3 years, you decide to make an extra payment of $\$2,000$. How much will this shorten your overall amortization period (in months)?

Problem # 20

Write a closed-form generating function for each of the following sequences:

a) $1, -1, 1, -1, 1, -1, \ldots$

b) $1, 0, 1, 0, 1, 0, \ldots$

c) $1, 1, 1, 1, 1, 1, \ldots$

d) $1, 1, -1, -1, 1, 1, -1, -1, \ldots$

e) $1, 0, -1, 0, 1, 0, -1, 0, \ldots$

f) $1, 0, 0, 1, 0, 0, 1, 0, 0, \ldots$

g) $1, 1, 0, 1, 1, 0, 1, 1, 0, \ldots$

h) $0, 0, 0, 1, 1, 1, 1, 1, 1, \ldots$

i) $1, 2, 3, 4, 5, 6, \ldots$

j) $1, 0, 2, 0, 3, 0, \ldots$

k) $1, -2, 3, -4, 5, -6, \ldots$

l) $1, 2, -3, -4, 5, 6, -7, -8. \ldots$

m) $1, 0, 3, 0, 5, 0, 7, 0, \ldots$

n) $1, 0, -3, 0, 5, 0, -7, 0, \ldots$

o) $1, 2, 0, 4, 5, 0, 7, 8, 0, \ldots$

p) $1, 2, 0, 3, 4, 0, 5, 6, 0, \ldots$

q) $1, 1, 2, 1, 3, 1, 4, 1, \ldots$

r) $1, 3, 6, 10, 15, 21, \ldots$

Problem # 21
The Fibonacci representation of a non-negative integer is its (unique) expression as a sum of distinct non-consecutive Fibonacci numbers. Compute the Fibonacci representation of each of the following numbers:
a) 202, b) 105, c) 128, d) 243.

Bonus Problem

Problem # 22
Write a program to compute the Fibonacci representation of an arbitrary positive integer.

Dr. A. Betten Fall 2008

MATH 501 Introduction to Combinatorial Theory

Assignment # 4

Problem # 23

a) Find the number of integers from 1 to $10,000$ (inclusively) which are not divisible by $4, 5$ or 6.

b) Find the number of integers from 1 to $10,000$ (inclusively) which are not divisible by $4, 6, 7$ or 10.

c) Find the number of integers from 1 to $10,000$ (inclusively) which are neither squares nor cubes.

d) How many integers from 0 to $99,999$ (inclusively) have among their digits each of $2, 5$ and 8?

Problem # 24

Figure out the definition of an equivalence relation on a set. Also, find out what a partial order is. Which of the following are equivalence relations / partial orders ? Explain.

a) A loves B (people).

b) A is taller than B (people).

c) Divisble by 3 (integers).

d) Having the same remainder after division by 3 (integers).

e) Having the same image under a given function f (real numbers).

f) A is a subset of B (sets).

Problem # 25

Figure out the definition of an involution.

a) Find a recurrence relation for the number s_n of involutions in Sym_n (the set of permutations of n things).

b) Prove that s_n is even for $n > 1$.

c) Show that $s_n > \sqrt{n!}$ for all $n > 1$.

Dr. A. Betten Fall 2008

MATH 501 Introduction to Combinatorial Theory

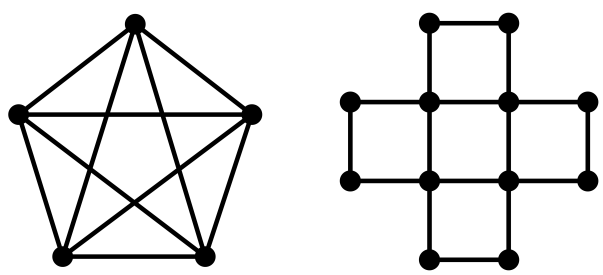
Assignment # 5

Problem # 26
Let $G = (V, E)$ be a graph. The line graph of G is the graph $L(G) = (E, F)$ where F consists of the pairs of edges if E that intersect. Find the line graph of K_4.

Problem # 27

a) An Euler cycle is a path through all the edges of a graph which starts and ends at the same vertex (you may visit the same vertex repeatedly but never go the same edge twice). How many Euler cycles do each of the following graphs have?

b) A Hamiltonian cycle is a cycle through all the vertices of the graph (you must visit every vertex exactly once). How many Hamiltonian cycles do each of the graphs have?



Problem # 28
Let G be a group. Let S be a subset of G. The Cayley graph (G, S) is the graph $\text{Cayley}(G, S) = (G, E)$ where x and y is joined by and edge in E if $xs = y$ or $ys = x$ for some s in S. Let $\mathbb{Z}_n$ be the additive group of integers modulo n.

a) Draw $\text{Cayley}(\mathbb{Z}_8, \{1, 2\})$

b) Draw $\text{Cayley}(\mathbb{Z}_8, \{1, 3\})$

c) Are the two graphs isomorphic?

Problem # 29
The hypercube graph Q_n is the Cayley graph of $\mathbb{F}_2^n$ with respect to $S = \{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$. Here, $\mathbf{e}_i$ is the i-th unit vector, i.e. the vector whose only nonzero entry is in the i-th position, this entry being one.

a) Draw Q_1, Q_2, Q_3.

b) Find a Hamiltonian cycle in Q_n.

Problem # 30
For an integer $n \geq 2$, consider the matrices

$$A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad B = \begin{pmatrix} \cos 2\pi/n & \sin 2\pi/n \\ \sin 2\pi/n & -\cos 2\pi/n \end{pmatrix}.$$

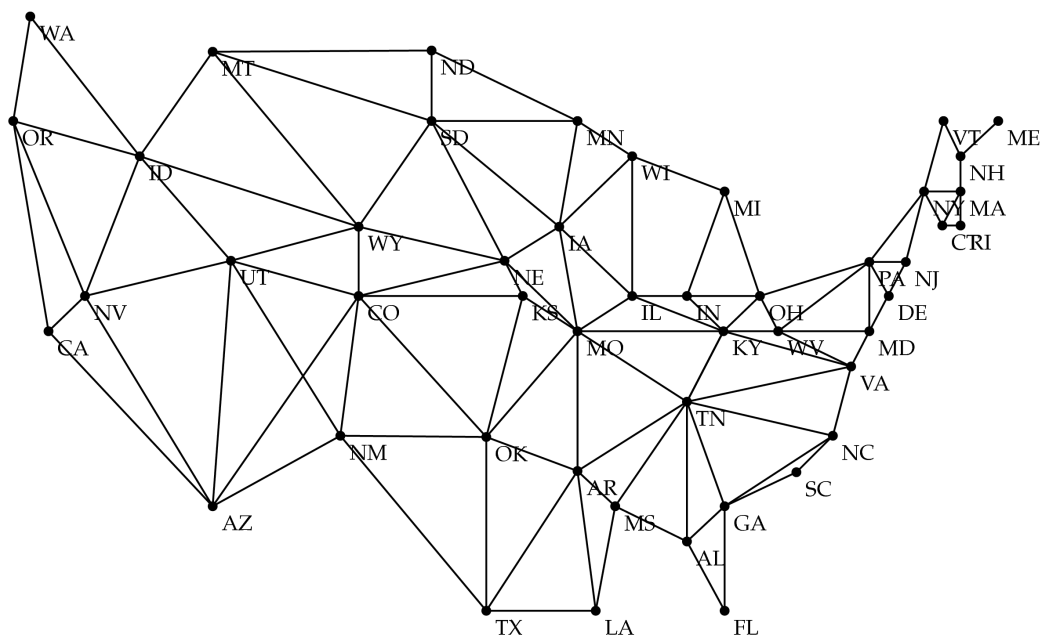
Consider the linear maps Φ and Ψ from $\mathbb{R}^2$ to $\mathbb{R}^2$ associated to these matrices.

a) What is the order of the group D_n generated by Φ and Ψ?

b) Let G be the Cayley graph of D_n with respect to Φ and Ψ. Find a Hamiltonian cycle in this graph G.

Problem # 31
Show all trees on 6 vertices (up to isomorphism).

Problem # 32
Here is a graph.



a) Explain which relation is represented by this graph. Look up the definition of "relation" if you are unsure about that.

b) Find out what a cut vertex is. Does this graph contain any?

c) Find the vertices of highest / lowest degree.

d) Find the shortest path from CA to ME and from SC to ND.

e) Find out what a planar graph is. Is this graph planar? Why or why not?

Bonus Problem

Find a 3-regular graph on 10 vertices with girth 5. A graph is k-regular if it has degree sequence $(k, k, \ldots, k)$; the girth is the length of the shortest cycle.

Dr. A. Betten Fall 2008

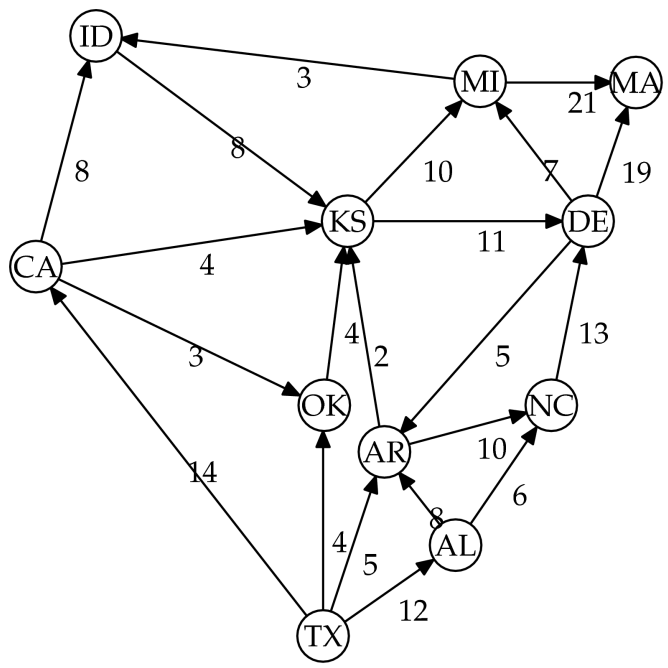
MATH 501 Introduction to Combinatorial Theory

Assignment # 6

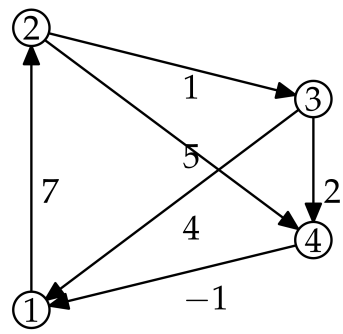
Problem # 33
Construct five nonisomorphic graphs with degree sequence $(5, 3, 2, 2, 1, 1, 1, 1)$. *Hint:* One of them is disconnected.

Problem # 34
Construct two nonisomorphic graphs with degree sequence $(1, 1, 2, 2, 3, 3)$.

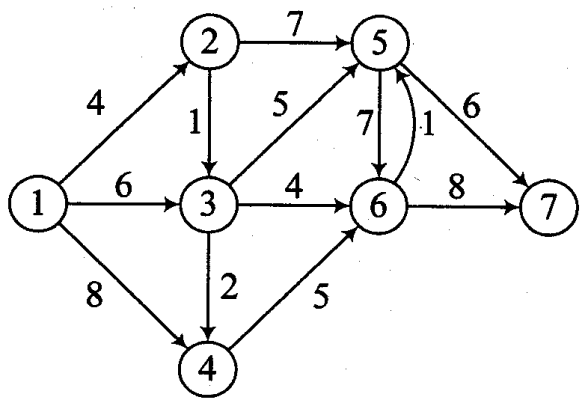
Problem # 35
In the following network, the edge labels represent capacities. Compute the maximum flow from TX to MA. Then exhibit a minimum cut.



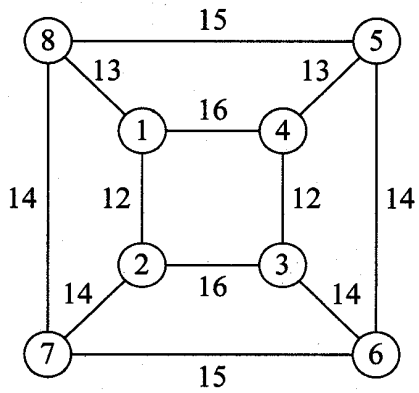
Problem # 36
Using the algorithm of Floyd-Warshall, compute all shortest distances in the following graph.



Problem # 37
Using Dijkstra's algorithm, compute the distances from vertex 1 in the following graph.



Problem # 38
Compute a minimum spanning tree for the following network.

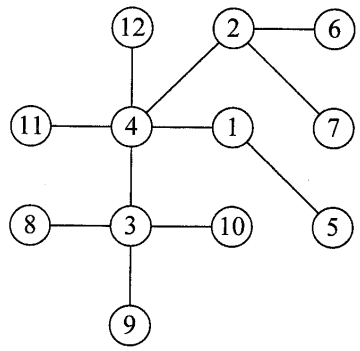


Dr. A. Betten Fall 2008

MATH 501 Introduction to Combinatorial Theory

Assignment # 7

Problem # 39
Compute the Prüfer code of the following tree:



Problem # 40
Compute the labeled tree with Prüfer sequence $(9, 8, 2, 3, 2, 1, 7, 7, 1)$.

Problem # 41
Show all trees on 6 vertices (up to isomorphism). For each tree, give the number of essentially different labelings. Use this to verify Cayley's Theorem for $n = 6$.

Problem # 42
Up to isomorphism, how many graphs are there on 4 vertices? A graph is self-complementary if it is isomorphic to its complement (where edges exist precisely if they did not exist in the original graph). How many self-complementary graphs are there on 4 vertices? (on 5 vertices?)

Problem # 43
Verify that $(3, 3, 2, 2, 1, 1, 1, 1)$ is the degree sequence of a tree. Construct three nonisomorphic trees with this degree sequence.

Problem # 44
The chromatic number of a graph is the least number of colors needed to assign to the vertices in such a way that all edges are between vertices of different color. Find the chromatic number of the n-cycle.

Bonus Problem

Problem # 45
Find a self-orthogonal Latin square of order 5 (You may write a computer program for this).

Dr. A. Betten Fall 2008

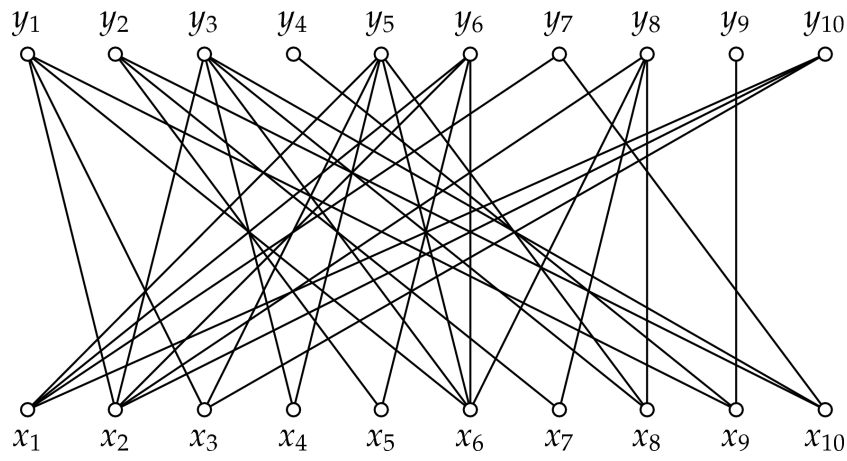
MATH 501 Introduction to Combinatorial Theory

Assignment # 8

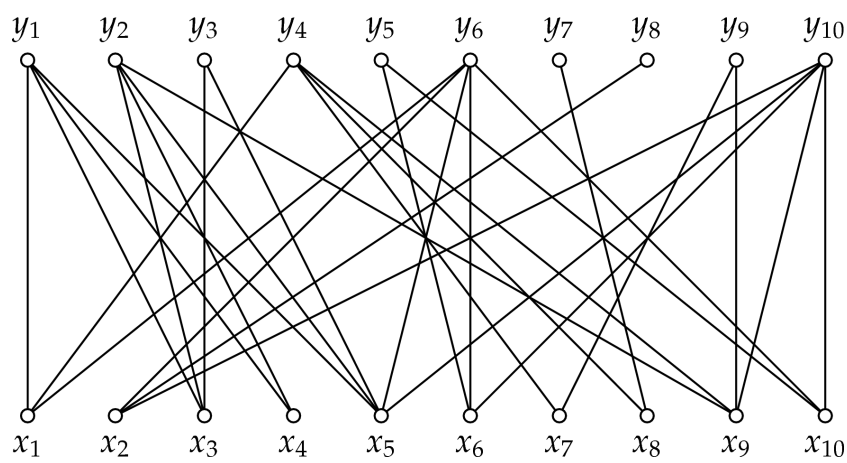
Problem # 46

For each of the following bipartite graphs, find a perfect matching or show that none exists. Use the algorithm from the lecture.

a)



b)



Problem # 47

An independent set in a graph G is a set S of vertices such that no two vertices in S are connected by an edge in G. What is the size of the largest independent set in the following graphs G?

a) $G = C_n$ the n-cycle.

b) $G = P_{n-1}$ the path of length $n-1$ (i.e., with n vertices).

Problem # 48
Consider the hypercube graph H_n. For vectors $\mathbf{x}$ and $\mathbf{y}$, let $d(\mathbf{x},\mathbf{y}) = |\{i \mid x_i \neq y_i\}|$ be the number of coordinates in which $\mathbf{x}$ and $\mathbf{y}$ differ. For $\mathbf{x} \in H_n$ and $m \in \mathbb{N}$, how many vectors $\mathbf{y} \in H_n$ satisfy $d(\mathbf{x},\mathbf{y}) \leq m$?

Problem # 49
We are using the same notation as in the previous exercise. Let $\mathbf{x}$ and $\mathbf{y}$ be two vectors with $d(\mathbf{x},\mathbf{y}) = d$. For $r, s \in \mathbb{N}$, let

$$t = |\{\mathbf{z} \in H_n \mid d(\mathbf{x},\mathbf{z}) = r \text{ and } d(\mathbf{y},\mathbf{z}) = s\}|.$$

Prove the following statements:

a) If $d + r - s \geq 0$ and $d + r - s$ is even then $t = \binom{d}{i}\binom{n-d}{r-i}$, where $i = \frac{d+r-s}{2}$.

b) If $d + r - s$ is odd or $d + r - s < 0$ then $t = 0$.

c) If $r + s = d$ then $t = \binom{d}{r}$.

d) H_n is strongly regular only for $n \leq 2$.

Problem # 50
Verify that the Paley graph $P(q)$ ($q \equiv 1 \bmod 4$, q an odd prime power) is strongly regular with $v = q$, $k = (q-1)/2$, $\lambda = (q-5)/4$, and $\mu = (q-1)/4$ by checking the equation

$$A^2 + (\mu - \lambda)A + (\mu - k)I = \mu J$$

where $A = \frac{1}{2}(Q + J - I)$ is the adjacency matrix. As usual, $Q = (q_{i,j})$ with $q_{i,j} = \chi(\gamma_i - \gamma_j)$, where $\mathbb{F}_q = \{\gamma_0, \gamma_1, \ldots, \gamma_{q-1}\}$. Also, J is the all-one matrix and I is the identity matrix. Last, $\chi : \mathbb{F}_q \to \mathbb{C}$ is the map

$$\chi(\gamma) = \begin{cases} 0 & \text{if} \quad \gamma = 0 \\ 1 & \text{if} \quad \gamma \in \square \\ -1 & \text{if} \quad \gamma \in \not\square. \end{cases}$$

Problem # 51
Let A be the adjacency matrix of a strongly regular graph $\mathrm{srg}(v, k, \lambda, \mu)$. Note that k is an eigenvalue of A. Use the quadratic equation

$$A^2 + (\mu - \lambda)A + (\mu - k)I = \mu J$$

to determine the other two eigenvalues r and s of A. It can be shown that the multiplicity of k as eigenvalue of A is one. Under this hypothesis, derive a formula for the multiplicities f and g of r and s, respectively.

Dr. A. Betten Fall 2008

MATH 501 Introduction to Combinatorial Theory

Assignment # 9

Take Home Final. Due date: 12/18/08.

Problem # 52

Evaluate the minimum distances of the binary codes generated by

$$\begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Problem # 53

A linear code C is self-orthogonal if and only if $\langle c, c' \rangle = 0$ for all $c, c' \in C$. Show that C is self-dual (i.e., $C = C^{\perp}$) if and only if C is self-orthogonal and C is of dimension $k = n/2$ (and hence n is even).

Problem # 54

Let C be a binary, self-orthogonal code.

a) Show that each word of C is even and that $C^{\perp}$ contains the all-one vector $\mathbf{1}$.

b) Assume in addition that the length n of C is odd and that the dimension of C is $(n-1)/2$. Show that

$$C^{\perp} = C \cup (\mathbf{1} + C).$$

Problem # 55

Show that a code with check matrix $H = (I_k \mid A)$ is self-dual if and only if A is a square matrix with $A \cdot A^{\top} = -I_k$.

Problem # 56

Define the "intersection" of two binary vectors u and v to be the vector

$$u \wedge v \ : \ (u_0 v_0, \ldots, u_{n-1} v_{n-1})$$

which has ones only where both u and v have ones. Also, let

$$u \vee v \ : \ (1 - (1 - u_0)(1 - v_0), \ldots, 1 - (1 - u_{n-1})(1 - v_{n-1}))$$

be the "union" of u and v, i.e. the vector which is one if at least one of u or v is one. Show that

$$\mathrm{wt}(u + v) = \mathrm{wt}(u) + \mathrm{wt}(v) - 2\mathrm{wt}(u \wedge v) = \mathrm{wt}(u \vee v) - \mathrm{wt}(u \wedge v).$$

Problem # 57

Show the following:

a) If $u, v \in \mathbb{F}_2^n$, then $\langle u, v\rangle \equiv \mathrm{wt}(u \wedge v) \bmod 2$ (where $u \wedge v$ is as in the previous problem).

b) If $u \in \mathbb{F}_2^n$, then $\langle u, u\rangle \equiv \mathrm{wt}(u) \bmod 2$.

c) If $u \in \mathbb{F}_3^n$, then $\langle u, u\rangle \equiv \mathrm{wt}(u) \bmod 3$.

Problem # 58
An $[n, k]_q$ code with minimum distance d is said to be perfect if the balls of radius $e = \lfloor (d-1)/q \rfloor$ cover the whole Hamming space $H(n, q)$. Show that this is equivalent to

$$\sum_{i=0}^{\lfloor (d-1)/2 \rfloor} \binom{n}{i} (q-1)^i = q^{n-k}.$$

Deduce that

$$\sum_{i=0}^{\lfloor (d-1)/2 \rfloor} \binom{n}{i} (q-1)^i \leq q^{n-k}$$

for any linear code. Is the binary $(7, 4)$-Hamming code perfect?

Problem # 59
How many one-dimensional subspaces does the vector space $\mathbb{F}_q^n = H(n, q)$ have?

Problem # 60
Let H be a matrix whose columns form a system of representatives of the one-dimensional subspaces of $\mathbb{F}_q^m$. The code whose check matrix is H is called m-th order q-ary Hamming code. What are its parameters? Is it a perfect code?

Problem # 61
Let $\mathcal{C}$ be a linear $[n, k]_q$ code with distance d. The parity extension of $\mathcal{C}$ is

$$P(\mathcal{C}) := \{(c_0, \ldots, c_{n-1}, c_n) \mid (c_0, \ldots, c_{n-1}) \in \mathcal{C}, c_n = -\sum_{i=0}^{n-1} c_i\}$$

Compute the minimum distance of $P(\mathcal{C})$ (Hint: distinguish cases according to whether d is even or odd).

Problem # 62
Compute the check matrix of the ternary code generated by

$$\begin{pmatrix} 1 & 0 & -1 & 1 & 0 & 1 \\ -1 & -1 & -1 & 0 & 1 & 1 \\ 0 & 1 & -1 & -1 & -1 & -1 \end{pmatrix}$$

Problem # 63
Explain why the Hamming distance is a metric.

Problem # 64
Can you make a Reed-Solomon code that can correct two errors? Can you make a generator and a check matrix for the code?

Problem # 65
Suppose you want to make a binary 2-error-correcting code of length 21. What dimension can you achieve?

Problem # 66
Suppose you want to make a BCH-code of length 17 over the field $\mathbb{F}_4$ with minimum distance at least 6. What dimension k can you achieve? Hint: Try using the cyclotomic sets containing 0, 1 and 2.