

M501 Combinatorics I

exercise sheet # 2

Exercise # 1

(3 points)

Prove the following identities:

$$\text{a) } \binom{n}{k} \binom{k}{l} = \binom{n}{l} \binom{n-l}{k-l},$$

$$\text{b) } \sum_{i=0}^k \binom{m}{i} \binom{n}{k-i} = \binom{m+n}{k} \quad (\text{recall that } \binom{n}{k} = 0 \text{ for } k < 0 \text{ or } k > n),$$

$$\text{c) } \sum_{i=0}^k \binom{n+i}{i} = \binom{n+k+1}{k},$$

$$\text{d) } \sum_{k=0}^n (-1)^k \binom{n}{k}^2 = \begin{cases} 0 & \text{if } n \text{ is odd} \\ (-1)^m \binom{2m}{m} & \text{if } n = 2m. \end{cases}$$

Exercise # 2

(3 points)

Let F_n be the n -th Fibonacci number. Prove the following identities:

$$\text{a) } F_n^2 - F_{n+1}F_{n-1} = (-1)^n \text{ for } n \geq 1.$$

$$\text{b) } \sum_{i=0}^n F_i = F_{n+2} - 1.$$

$$\text{c) } F_{n-1}^2 + F_n^2 = F_{2n}, \quad F_{n-1}F_n + F_nF_{n+1} = F_{2n+1}.$$

$$\text{d) } F_n = \sum_{i=0}^{\lfloor n/2 \rfloor} \binom{n-i}{i}.$$

Exercise # 3

(1 points)

Let F_n be the n -th Fibonacci number. Show that F_n is composite for all odd $n > 3$.

Exercise # 4

(3 points)

If n is a multiple of 8, show that the number of subsets of an n -set of size divisible by 4 is $2^{n-2} + 2^{(n-2)/2}$. Example: for $n = 8$, we have $\binom{8}{0} + \binom{8}{4} + \binom{8}{8} = 1 + 70 + 1 = 72 = 64 + 8 = 2^6 + 2^3$.

Hint: consider $(1+i)^n$, where $i = \sqrt{-1}$.

due to Friday, 9/23/05.