

**M501 Combinatorics I**

## exercise sheet # 1

**Exercise # 1** (3 points)

Prove: The infinite series  $\sum_{j \geq 0} F_j(x)$  converges if and only if  $\lim_{j \rightarrow \infty} \deg F_j(x) = \infty$ .

**Exercise # 2** (3 points)

Prove: The infinite product  $\prod_{j \geq 1} (1 + F_j(x))$ , where  $F_j(0) = 0$ , converges if and only if  $\lim_{j \rightarrow \infty} \deg F_j(x) = \infty$ .

**Exercise # 3** (1 points)

A polynominoe of order  $n$  is a arrangement of  $n$  identical squares in the  $\mathbb{Z} \times \mathbb{Z}$  grid which is connected in the sense that one can get from any square to any other square by crossing only along edges which are adjacent to two squares. Also, the arrangement is required to be “horizontally convex,” meaning that the squares in horizontal rows form one interval (i.e., no gaps in between them). Determine all polynominoes with  $n \leq 4$  squares. Hint: there are 19 polynominoes with 4 squares.

Bonus (hard): Determine the generating function for the number of polynominoes of order  $n$ .

**Exercise # 4** (3 points)

In the  $\mathbb{Z} \times \mathbb{Z}$  plane, one starts from  $(0, 0)$  and makes moves either  $(1, 0)$  (i.e. East or E) or  $(-1, 0)$  (i.e., West or W) or  $(0, 1)$  (i.e. North or N). Let  $f(n)$  be the number of non self-intersecting paths which are made up of  $n$  such moves. In other words,  $f(n)$  counts the strings of length  $n$  over the letters  $E, W, N$  which do not contain  $EW$  or  $WE$  as a substring. Determine  $f(n)$ . Do this by finding a simple expression for the generating function.

due to Friday, 9/2/05.