

M301 Introduction to Combinatorial Theory

homework sheet # 1

The pigeonhole principle:

- a) If n pigeonholes shelter $n + 1$ or more pigeons, at least 1 pigeonhole shelters at least 2 pigeons.
- b) If n pigeonholes shelter $kn + 1$ pigeons, where k is a positive integer, at least 1 pigeonhole shelters at least $k + 1$ pigeons.
- c) If there are $p_1 + p_2 + \cdots + p_n - n + 1$ or more pigeons among hole 1, hole 2, \dots , and hole n , then, for some j , hole j has p_j or more pigeons.
- d) Given m pigeons within n pigeonholes, at least 1 hole contains $p + 1$ pigeons, where $p = \lfloor (m - 1)/n \rfloor$.

Problem # 1

Prove all pigeonhole principles.

Problem # 2

A box contains 20 cell phones, of which 4 are Nokia, 7 are Motorola and 9 are Samsung. What is the smallest number of cell phones which must be chosen (blindfolded) so that the selection is guaranteed to contain $r = 4, 5, 6, 7, 8, 9$ phones of the same make?

Problem # 3

If an undergraduate club is supposed to include at least 6 freshmen or at least 5 sophomores or at least 4 juniors or at least 3 seniors, what is the least number of students who are needed to join to meet the condition regardless of how the students are selected.

Problem # 4

Suppose there are 26 students and 7 cars to transport them. Show that at least one car has 4 passengers.

Problem # 5

How many among the first 100000 positive integers contain exactly one 3, one 4, and one 5 in their decimal representation?

Problem # 6

Find the total number of positive integers with distinct digits.

Problem # 7

If a set X has $2n + 1$ elements, find the number of subsets of X with at most n elements.

Problem # 8

Find the number of positive integers with n digits in which no two adjacent digits are the same. How many of these are even?

Problem # 9

Find the number of n -character strings that can be formed using the letters A, B, C, D and E such that each string has an even number of A 's.

Problem # 10

Prove that $\binom{p+q}{r} = \sum_{j=0}^r \binom{p}{j} \binom{q}{r-j}$.

Problem # 11

Prove that $\binom{n}{r} \binom{r}{k} = \binom{n}{k} \binom{n-k}{r-k}$.

Problem # 12

Show that if p is a prime number, $\binom{p}{r}$ is divisible by p for $r = 1, 2, \dots, p-1$.

Problem # 13

Find the number of (distinct!) words that can be formed from the letters of the words (a) ELASTIC and (b) ASBESTOS. (We are using all the letters.)

Problem # 14

Consider a round robin tournament (i.e. a tournament in which each player plays against every other player once). Assume that each player wins at least once. Show that there are at least 2 players having the same number of wins.

Problem # 15

Show that any set of n integers has a subset such that the sum of the integers in the subset is divisible by n .

Problem # 16

Smalltown USA has a town council consisting of 10 Democrats and 11 Republicans (and no others). There are 4 women among the Democrats and 3 women among the Republicans. Find the number of planning committees of 8 councillors which have equal numbers of men and women and equal numbers from both parties.

Problem # 17

A box contains 7 balls: blue one, blue two, blue three, red one, red two, green one and green two. Three of the balls are to be selected at random. How many such subsets contain at least two blue balls?

Problem # 18

Let us (temporarily) call all positive integers of the form $6k + 1$ ($k \in \mathbb{N}$) "awesome" numbers. Show that the product of awesome numbers is awesome. Show that the sum of seven awesome numbers is awesome. Under what condition is the sum of n awesome numbers awesome?

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