Combinatorics and Computer Algebra (CoCoA 2015)

Combinatorics, Modular Functions, and Computer Algebra

Peter Paule
(joint work with: G.E. Andrews, S. Radu)

Johannes Kepler University Linz
Research Institute for Symbolic Computation (RISC)
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Example: \( p(4) = 5 \): 4, 3 + 1, 2 + 2, 2 + 1 + 1, 1 + 1 + 1 + 1.
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**EULER.** The generating function of the partition numbers is

\[
E(q) := \sum_{n=0}^{\infty} p(n)q^n = \prod_{n=1}^{\infty} \frac{1}{1 - q^n}
\]

\[
= (1 + q^1 + q^{1+1} + q^{1+1+1} + \ldots) \\
\times (1 + q^2 + q^{2+2} + q^{2+2+2} + \ldots) \\
\times (1 + q^3 + q^{3+3} + q^{3+3+3} + \ldots) \\
\times \text{ etc.} \\
= \ldots + q^{1+1+1+2+2+3+\ldots} + \ldots
\]
Back to $p(n)$: ANY PATTERNS?

- 3,313, 3,325, and 3,362 of the first 10,000 entries are congruent respectively to 0, 1, and 2 modulo 3. [Ahlgren & Ono]

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WHY? Let’s look again at our table:
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Ramanujan’s famous congruences

\[
p(5n + 4) \equiv 0 \pmod{5},
p(7n + 5) \equiv 0 \pmod{7},
p(11n + 6) \equiv 0 \pmod{11}
\]
Proof. The first congruence is implied by

\[ \sum_{n=0}^{\infty} p(5n + 4)q^n = 5 \prod_{j=1}^{\infty} \frac{(1 - q^{5j})^5}{(1 - q^j)^6} \]
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\]

“It would be difficult to find more beautiful formulae than the ‘Rogers-Ramanujan’ identities . . .; but here Ramanujan must take second place to Prof. Rogers; and, if I had to select one formula from all Ramanujan’s work, I would agree with Major MacMahon in selecting . . .” [G.H. Hardy]
Ramanujan [1919] found also an identity for the 7-case:

\[
\sum_{n=0}^{\infty} p(7n + 5)q^n = 7 \prod_{j=1}^{\infty} \frac{(1 - q^{7j})^3}{(1 - q^j)^4} + 49q \prod_{j=1}^{\infty} \frac{(1 - q^{7j})^7}{(1 - q^j)^8}.
\]

What about the case \( p(11n + 6) \)?
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What about the case \( p(11n + 6) \)?

No such identity has been known, until recently . . .
Modular Functions
Recall

\[ E(q) := \prod_{n=1}^{\infty} (1 - q^n). \]

We move from formal power (resp. Laurent) series to complex functions by defining

\[ q := q(\tau) := \exp(2\pi i \tau)(= e^{2\pi i \tau}), \]

where here and in the following

\[ \tau \in \mathbb{H} := \{ z \in \mathbb{C} : \text{Im}(z) > 0 \}. \]
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---

**MODULAR SYMMETRY.** If \( g(\tau) := q(\tau) E(q(\tau))^24 \) then

\[ g \left( \frac{a\tau + b}{c\tau + d} \right) = (c\tau + d)^{12} g(\tau) \text{ for all } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z}). \]
We also need subgroups where \( N \in \mathbb{N} \setminus \{0\} \):

\[
\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z}) : N | c \right\}.
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**Def.** A holomorphic function \( f(\tau) \) is a modular function for \( \Gamma_0(N) \), in short: \( f \in M(N) \), if

\[
f\left( \frac{a\tau + b}{c\tau + d} \right) = f(\tau) \quad \text{for all} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)
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and for all $p/q \in \mathbb{Q} \cup \{\infty\}$:

$$\lim_{\tau \to p/q} |f(\tau)| \in \mathbb{R}_{\geq 0} \cup \{\infty\}.$$
Examples. Eta quotients like

\[ q \frac{E(q^5)^6}{E(q)^6} = \prod_{j=1}^{\infty} \frac{(1 - q^5j)^6}{(1 - q^j)^6} \in M(5), \]

\[ q \frac{E(q^7)^4}{E(q)^4} = \prod_{j=1}^{\infty} \frac{(1 - q^7j)^4}{(1 - q^j)^4} \in M(7); \]

functions like

\[ q E(q^5) \sum_{n=0}^{\infty} p(5n + 4)q^n \in M(5), \]

or the celebrated modular \( j \)-function (Klein invariant),

\[ j(\tau) = \frac{1}{q} + 744 + 196884q + 21493760q^2 + 864299970q^3 + \cdots \in M(1). \]

NOTE. \( \Gamma_0(1) = \text{SL}_2(\mathbb{Z}). \)
Zero recognition of functions in $M(N)$?

The action

$$\Gamma_0(N) \times \mathbb{H} \rightarrow \mathbb{H}, \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \tau := \frac{a\tau + b}{c\tau + d}$$

induces the action

$$\Gamma_0(N) \times \text{HoloFus}(\mathbb{H}) \rightarrow \text{HoloFus}(\mathbb{H}), (f|\gamma)(\tau) := f(\gamma \tau).$$
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The modular functions in $M(N) \subseteq \text{HoloFus}(\mathbb{H})$ are those which are constant on the orbits of the first action. It is crucial to extend this action as follows:
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$$

$$
X_0(N) := \{[\tau] : \tau \in \mathbb{H} \cup \mathbb{Q} \cup \{\infty\}\} \quad \text{(set of orbits)}.
$$
The set of orbits

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can be turned into a **COMPACT** Riemann surface.
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Each (holomorphic) \( f \in M(N) \) has a **meromorphic** extension

\[ \tilde{f} : X_0(N) \to \mathbb{C} \cup \{ 0 \}. \]

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In our context we can restrict to “good” subalgebras of \( M(N) \):
To prove (and to find!) Ramanujan’s identities automatically, we can restrict to consider

\[ M^\infty(N) := \{ f \in M(N) : \tilde{f} \text{ has a pole at } [\infty] \}. \]
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Our goal is to present a given \( f \in M^\infty(N) \) in the form

\[
f = c_1 \frac{E(q^{a_1})^{\alpha_1} E(q^{a_2})^{\alpha_2} \cdots}{E(q^{b_1})^{\beta_1} E(q^{b_2})^{\beta_2} \cdots} + \cdots + c_m \frac{E(q^{u_1})^{\mu_1} E(q^{u_2})^{\mu_2} \cdots}{E(q^{v_1})^{\nu_1} E(q^{v_2})^{\nu_2} \cdots}
\]

with coefficients \( c_j \in \mathbb{C} \) and the Eta quotients in \( M^\infty(N) \).
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with coefficients \( c_j \in \mathbb{C} \) and the Eta quotients in \( M^\infty(N) \).

All Eta quotients in \( M^\infty(N) \) form a multiplicative monoid denoted by

\[ E^\infty(N) := \{ f \in M^\infty(N) : f \text{ is an Eta quotient} \} \]
NOTE 1. This monoid is finitely generated,

\[ E^\infty(N) = \langle f_1, \ldots, f_n \rangle = \{ f_1^{\ell_1} \cdots f_n^{\ell_n} : \ell_i \in \mathbb{N} \}; \]

the generators \( f_1, \ldots, f_n \in E^\infty(N) \) for fixed \( N \) can be determined algorithmically (e.g., with partition analysis).
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NOTE 2. In view of

\[ f = c_1 \frac{E(q^{a_1})^{\alpha_1} E(q^{a_2})^{\alpha_2} \cdots}{E(q^{b_1})^{\beta_1} E(q^{b_2})^{\beta_2} \cdots} + \cdots + c_m \frac{E(q^{u_1})^{\mu_1} E(q^{u_2})^{\mu_2} \cdots}{E(q^{v_1})^{\nu_1} E(q^{v_2})^{\nu_2} \cdots} \]

our task is to constructively decide subalgebra membership

\[ f \in \mathbb{C}[f_1, \ldots, f_n] \subseteq M^\infty(N). \]
Monoids & Subalgebras
McNuggets Partitions

GIVEN

TASK: buy exactly 43 nuggets.
McNuggets Partitions

GIVEN

TASK: buy exactly 43 nuggets.

IMPOSSIBLE!
McNuggets Partitions

GIVEN

TASK: buy exactly 43 nuggets.

IMPOSSIBLE!

See: “How to order 43 Chicken McNuggets - Numberphile”
www.youtube.com/watch?v=vNTSugyS038
Generators of (additive) submonoids of $\mathbb{N}$

$M:=\langle 6, 9, 20 \rangle \subseteq \mathbb{N} = \{0, 1, \ldots \}$

$[M]_i := \{x \in M : x \equiv i \pmod{6} \}$
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$[M]_0 = \{0, 6, 12, \ldots \}$,
$[M]_1 = \{1, 7, 13, 19, 25, 31, 37, 43, 49, 54, \ldots \}$,
$[M]_2 = \{2, 8, 14, 20, 26, \ldots \}$,
$[M]_3 = \{3, 9, 15, \ldots \}$,
$[M]_4 = \{4, 10, 16, 22, 28, 34, 40, 46, \ldots \}$,
$[M]_5 = \{5, 11, 17, 23, 29, 35, \ldots \}$. 
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\[
M = \langle 6, 9, 20 \rangle = \langle 6, 49, 20, 9, 40, 29 \rangle \quad \text{and}
\]

\[
\{u_1, u_2, u_3, u_4, u_5\} := \{49, 20, 9, 40, 29\} \equiv \{1, 2, 3, 4, 5\} \pmod{6}.
\]
QUESTION. What happens if we add more generators?

\[ M^+ := \langle 4, 6, 9, 20 \rangle \subseteq \mathbb{N} = \{0, 1, \ldots\} \]

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\[ [M^+]_i := \{ x \in M : x \equiv i \pmod{6} \} \]

\[ [M^+]_0 = \{0, 6, 12, \ldots \}, \]
\[ [M^+]_1 = \{1, 7, 13, 19, \ldots \}, \]
\[ [M^+]_2 = \{2, 8, 14, \ldots \}, \]
\[ [M^+]_3 = \{3, 9, 15, \ldots \}, \]
\[ [M^+]_4 = \{4, 10, \ldots \}, \]
\[ [M^+]_5 = \{5, 11, 17, 23, \ldots \}. \]

NOTE. In contrast to the Frobenius number \( F(6, 9, 20) = 43 \), now \( F(4, 6, 9, 20) = 11 \) and

\[
43 = 2 \begin{array}{c} 4 \end{array} + 6 + 9 + 1 \begin{array}{c} 20 \end{array} \\
= \begin{array}{c} 4 \end{array} + 2 \begin{array}{c} 6 \end{array} + 3 \begin{array}{c} 9 \end{array} = [\text{four more}].
\]

QUESTION. How to find presentations in terms of generators?
(a) Greedy Algorithm

**STRATEGY:** Successive subtraction; begin with the largest generator, first taken a maximal number of times; iterate.

\[ x = 62 \text{ and } M = \langle 6, 9, 20 \rangle. \]

\[ 62 - 3 \times 20 = 2 \]
\[ 2 - 2 \times 9 = 4 \]
\[ 4 - 1 \times 6 = 1 \]
\[ 1 - 3 \times 6 = 4 \]
\[ 42 - 1 \times 20 = 42 \]
\[ 42 - 4 \times 9 = 6 \]
\[ 6 - 1 \times 6 = 0 \]
(a) Greedy Algorithm

STRATEGY: Successive subtraction; begin with the largest generator, first taken a maximal number of times; iterate.

EXAMPLE: $x = 62$ and $M = \langle 6, 9, 20 \rangle$.

$62 - 3\cdot 20 = 2 \checkmark$

$62 - 2\cdot 20 = 22,$

$22 - 2\cdot 9 = 4 \checkmark$
$22 - 1\cdot 9 = 13,$

$13 - 2\cdot 6 = 1 \checkmark$
$22 - 3\cdot 6 = 4 \checkmark$

$62 - 1\cdot 20 = 42,$

$42 - 4\cdot 9 = 6,$

$6 - 1\cdot 6 = 0 \checkmark$. 
SOLUTION. \(62 = 1 \ 6 + 4 \ 9 + 1 \ 20\).
SOLUTION. $62 = 1 \begin{array}{c} 6 \\ + \end{array} + 4 \begin{array}{c} 9 \\ + \end{array} + 1 \begin{array}{c} 20 \end{array}$.

Also,

$62 = 4 \begin{array}{c} 6 \\ + \end{array} + 2 \begin{array}{c} 9 \\ + \end{array} + 1 \begin{array}{c} 20 \end{array}$

and

$62 = 7 \begin{array}{c} 6 \\ + \end{array} + 0 \begin{array}{c} 9 \\ + \end{array} + 1 \begin{array}{c} 20 \end{array}$. 
SOLUTION. $62 = 1 \text{ 6} + 4 \text{ 9} + 1 \text{ 20}$. 

Also, 

$62 = 4 \text{ 6} + 2 \text{ 9} + 1 \text{ 20}$

and 

$62 = 7 \text{ 6} + 0 \text{ 9} + 1 \text{ 20}$. 

PREVIEW. The Omega package computes 

```
In[2]:= OEqR[
   OEqSum[x^a y^b z^c, \{6a + 9b + 20c = 62\}, \lambda ]
]
```

Out[2] = $(x^7 + x^4 y^2 + xy^4) z$.
(b) Algorithm based on the generators $u_1, \ldots, u_5$

STRATEGY: Find a representation $x = u_i + \ell \cdot 6$.
(b) Algorithm based on the generators \( u_1, \ldots, u_5 \)

**STRATEGY:** Find a representation \( x = u_i + \ell \cdot 6 \).

**EXAMPLE:** \( M = \langle 6, u_1, u_2, u_3, u_4, u_5 \rangle = \langle 6, 43, 20, 9, 40, 29 \rangle \).

and \( x = 62 \).
(b) Algorithm based on the generators $u_1, \ldots, u_5$

**STRATEGY:** Find a representation $x = u_i + \ell \langle 6 \rangle$.

**EXAMPLE:** $M = \langle 6, u_1, u_2, u_3, u_4, u_5 \rangle = \langle 6, 43, 20, 9, 40, 29 \rangle$.
and $x = 62$.

$62 \equiv 2 \pmod{6}$; hence we recall that

$$[M]_2 = \{ \emptyset, 6, 14, 20, 26, \ldots \};$$

consequently,

$$62 = 20 + 7 \langle 6 \rangle = u_2 + 7 \langle 6 \rangle.$$
Application: $\mathbb{C}$-subalgebras of $\mathbb{C}[z]$

**GIVEN:** $f_1, f_2, f_3 \in \mathbb{C}[z]$ with

\[
\begin{align*}
    f_1 &= z^6 + az^5 + \ldots, \\
    f_2 &= z^9 + bz^8 + \ldots, \\
    f_3 &= z^{20} + cz^{19} + \ldots;
\end{align*}
\]

**FIND:** $g_1, \ldots, g_k \in \mathbb{C}[z]$ such that

\[
\begin{align*}
    \mathbb{C}[f_1, f_2, f_3] &= \langle g_1, \ldots, g_k \rangle_{\mathbb{C}[t]} \\
    &= \mathbb{C}[t] + \mathbb{C}[t] g_1 + \cdots + \mathbb{C}[t] g_k.
\end{align*}
\]
Application: \( \mathbb{C} \)-subalgebras of \( \mathbb{C}[z] \)

GIVEN: \( f_1, f_2, f_3 \in \mathbb{C}[z] \) with

\[
\begin{align*}
    t := f_1 & = z^6 + a z^5 + \ldots, \\
    f_2 & = z^9 + b z^8 + \ldots, \\
    f_3 & = z^{20} + c z^{19} + \ldots;
\end{align*}
\]

FIND: \( g_1, \ldots, g_k \in \mathbb{C}[z] \) such that

\[
\mathbb{C}[f_1, f_2, f_3] = \langle g_1, \ldots, g_k \rangle_{\mathbb{C}[t]} = \mathbb{C}[t] + \mathbb{C}[t] g_1 + \cdots + \mathbb{C}[t] g_k.
\]

KEY IDEA:

\[
z^n + \ldots = z^{u_i+\ell} \overline{6} + \ldots = (z^{a_i \overline{9} + b_i \overline{20}} + \ldots) t^\ell + \ldots.
\]
DECISION PROCEDURE: $\mathbb{C}$-subalgebra membership

GIVEN: $f$ and $f_1, \ldots, f_n$ from $\mathbb{C}[z]$;

FIND: polynomials $p_0, p_1, \ldots, p_k \in \mathbb{C}[z]$ such that

$$f = p_0(t) + p_1(t)g_1 + \cdots + p_k(t)g_k$$

$$\in \langle 1, g_1, \ldots, g_k \rangle_{\mathbb{C}[t]} = \mathbb{C}[f_1, \ldots, f_n]$$

APPLICATION.
DECISION PROCEDURE: $\mathbb{C}$-subalgebra membership

GIVEN: $f$ and $f_1, \ldots, f_n$ from $\mathbb{C}[z]$;

FIND: polynomials $p_0, p_1, \ldots, p_k \in \mathbb{C}[z]$ such that

$$f = p_0(t) + p_1(t)g_1 + \cdots + p_k(t)g_k$$

$$\in \langle 1, g_1, \ldots, g_k \rangle_{\mathbb{C}[t]} = \mathbb{C}[f_1, \ldots, f_n]$$

APPLICATION. With this procedure the right hand sides of Ramanujan identities can be found automatically: just apply it to given $f \in M^\infty(N)$ and $f_1, \ldots, f_n \in E^\infty(N)$. In this context, $z = \frac{1}{q}$ because of possible poles sitting at $[\infty]$; the $f_j$ can be treated as formal Laurent series like

$$f_1 = \frac{a_6}{q^6} + \frac{a_5}{q^5} + \cdots,$$

$$f_2 = \frac{b_9}{q^9} + \frac{b_8}{q^8} + \cdots,$$

$$f_3 = \frac{c_{20}}{q^{20}} + \frac{c_{19}}{q^{19}} + \cdots,$$ etc.
Example. Radu’s “Ramanujan-Kolberg” package in STEP 1 computes that:

\[ q^{-1} \frac{E(q)^8}{E(q^7)^7} \sum_{n=0}^{\infty} p(7n + 5)q^n \in M^\infty(7). \]
Example. Radu’s “Ramanujan-Kolberg” package in STEP 1 computes that:

\[
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\]

In STEP 2 Radu’s package determines that the monoid is generated by one element (recall \(f_1 = t\)):

\[
E^\infty(7) = \langle f_1 \rangle = \langle q^{-1} \frac{E(q)^4}{E(q^7)^4} \rangle.
\]
Example. Radu’s “Ramanujan-Kolberg” package in STEP 1 computes that:

\[
q^{-1} \frac{E(q)^8}{E(q^7)^7} \sum_{n=0}^{\infty} p(7n + 5)q^n \in M_\infty(7).
\]

In STEP 2 Radu’s package determines that the monoid is generated by one element (recall \(f_1 = t\)):

\[
E^\infty(7) = \langle f_1 \rangle = \langle q^{-1} \frac{E(q)^4}{E(q^7)^4} \rangle.
\]

In STEP 3 the package determines the module presentation

\[
\mathbb{C}[f_1, \ldots, f_n] = \mathbb{C}[f_1] = \langle 1, g_1, \ldots, g_k \rangle \mathbb{C}[t] = \langle 1 \rangle \mathbb{C}[f_1].
\]
Example. Radu’s “Ramanujan-Kolberg” package in STEP 1 computes that:

\[ q^{-1} \frac{E(q)^8}{E(q^7)^7} \sum_{n=0}^{\infty} p(7n + 5)q^n \in M^\infty(7). \]

In STEP 2 Radu’s package determines that the monoid is generated by one element (recall \( f_1 = t \)):

\[ E^\infty(7) = \langle f_1 \rangle = \langle q^{-1} \frac{E(q)^4}{E(q^7)^4} \rangle. \]

In STEP 3 the package determines the module presentation

\[ \mathbb{C}[f_1, \ldots, f_n] = \mathbb{C}[f_1] = \langle 1, g_1, \ldots, g_k \rangle_{\mathbb{C}[t]} = \langle 1 \rangle_{\mathbb{C}[f_1]}. \]

In STEP 4 the package finds Ramanujan’s identity:

\[ q^{-1} \frac{E(q)^8}{E(q^7)^7} \sum_{n=0}^{\infty} p(7n + 5)q^n = 49 \cdot 1 + 7 \cdot f_1. \]
QUESTION. How to compute the $u_1, \ldots, u_5$ from $M = \langle 6, 9, 20 \rangle$?

ANSWER. To find $u_1 = 49$, we ask Omega to compute

\begin{verbatim}
In[3]:= OEqR[
    OEqSum[ x^a y^b z^c, \{-6a + 9b + 20c = 1\}, \lambda ]
]
\end{verbatim}

\begin{verbatim}
Out[3]= \frac{x^8 y z^2}{(1-x^3 y^2)(1-x^{10} z^3)}.
\end{verbatim}
Out[3] means that

\[
\sum_{a, b, c \geq 0} x^a y^b z^c = \frac{x^8 y z^2}{(1 - x^3 y^2) (1 - x^{10} z^3)};
\]

in other words,

\[
\binom{8}{1} + \mathbb{N} \binom{3}{2} + \mathbb{N} \binom{10}{0} + \mathbb{N} \binom{10}{3}
\]

is the solution set of \(-6a + 9b + 20c = 1\). For example,

\[
1 \bigcirc 9 + 2 \bigcirc 20 = 1 + 8 \bigcirc 6 = 49 = u_1.
\]

Analogously one computes \(u_2, \ldots, u_5\).
Partition Analysis
PROBLEM [Polya]. Consider triangles with sides of integer length such that

\[ 1 \leq \ell \leq m \leq n; \]

TASK: express the total number of such triangles in terms of \( n \).
PROBLEM [Polya]. Consider triangles with sides of integer length such that
\[ 1 \leq \ell \leq m \leq n; \]

TASK: express the total number of such triangles in terms of \( n \).

EXAMPLE. \( n = 3 \)
PROBLEM [folklore]. Consider triangles with sides of integer length such that
\[1 \leq a \leq b \leq c;\]

TASK: find the total number of all such triangles with perimeter \(n\).
PROBLEM [folklore]. Consider triangles with sides of integer length such that

\[ 1 \leq a \leq b \leq c; \]

TASK: find the total number of all such triangles with perimeter \( n \).

EXAMPLE. \( n = 9 \)
**SOLUTION IDEA.** Determine the generating function

\[ T(x, y, z; q) := \sum_{c \geq b \geq a \geq 1, \text{s.t. } a+b>c} x^a y^b z^c q^{a+b+c}. \]

Then all **Polya triangles** are given by the coefficient of \( z^n \) in \( T(x, y, z; 1) \),

and the total number of such triangles as the coefficient of \( z^n \) in \( T(1, 1, z; 1) \).
SOLUTION IDEA. Determine the generating function

\[ T(x, y, z; q) := \sum_{c \geq b \geq a \geq 1, \text{ s.t. } a+b>c} x^a y^b z^c q^{a+b+c}. \]

Then all triangles with given perimeter \( n \) are given by the coefficient of \( q^n \) in \( T(x, y, z; q) \),

and the total number of such triangles as the coefficient of \( q^n \) in \( T(1, 1, 1; q) \).
HOW TO FIND A SUITABLE REPRESENTATION OF

\[ T(x, y, z; q) := \sum_{c \geq b \geq a \geq 1, \text{s.t. } a + b > c} x^a y^b z^c q^{a+b+c} \]

Pick up the Omega package, freely available at www.risc.jku.at/research/combinat/software and do the following:

\texttt{ln[1]:= \texttt{\textless\textless RISC`Omega\textgreater\textgreater}
HOW TO FIND A SUITABLE REPRESENTATION OF

\[ T(x, y, z; q) := \sum_{c \geq b \geq a \geq 1, \text{s.t. } a + b > c} x^a y^b z^c q^{a+b+c} \]

Pick up the Omega package, freely available at www.risc.jku.at/research/combinat/software and do the following:

\texttt{ln[1]:= << RISC\`Omega`}

Omega Package by Axel Riese (in cooperation with George E. Andrews and Peter Paule) - ©RISC, JKU Linz - V 2.47
HOW TO FIND A SUITABLE REPRESENTATION OF

\[ T(x, y, z; q) := \sum_{c \geq b \geq a \geq 1, \text{s.t. } a + b > c} x^a y^b z^c q^{a+b+c} \]

Pick up the Omega package, freely available at
www.risc.jku.at/research/combinat/software
and do the following:

\[
\text{In[1]} := \langle\langle \text{RISC`Omega``} \\
\text{In[2]} := \text{OR[OSum}[x^a y^b z^c q^{a+b+c}, \\
\{a \geq 1, b \geq a, c \geq b, a + b > c\}, 1]]
\]
\texttt{In[2]:= OR[OSum[\text{Sum}[x^a y^b z^c q^{a+b+c},
\{a \geq 1, b \geq a, c \geq b, a+b > c\}], l]]}

Assuming $b \geq 0$

Assuming $c \geq 0$

Eliminating $l_2$...

Eliminating $l_3$...

Eliminating $l_1$...

\texttt{Out[2]=} \quad \frac{q^3 x y z}{(1-q^2 y z)(1-q^3 x y z)(1-q^4 x y z^2)}
\begin{align*}
\text{In}[2]:= & \quad \text{OR} \left[ \text{OSum} \left[ x^a y^b z^c q^{a+b+c}, \right. \\
& \quad \left. \{a \geq 1, b \geq a, c \geq b, a + b > c\}, 1 \right]\right] \\
\text{Assuming} & \quad b \geq 0 \\
\text{Assuming} & \quad c \geq 0 \\
\text{Eliminating} & \quad l_2 \ldots \\
\text{Eliminating} & \quad l_3 \ldots \\
\text{Eliminating} & \quad l_1 \ldots \\
\text{Out}[2]= & \quad \frac{q^3 x y z}{(1 - q^2 y z) (1 - q^3 x y z) (1 - q^4 x y z^2)}
\end{align*}

WHAT STANDS BEHIND THIS?
MacMahon’s Partition Analysis
“The no. of partitions of \( N \) of the form \( N = b_1 + \cdots + b_n \) satisfying

\[
\frac{b_n}{n} \geq \frac{b_{n-1}}{n-1} \geq \cdots \geq \frac{b_2}{2} \geq \frac{b_1}{1} \geq 0
\]

equals the no. of partitions of \( N \) into odd parts each \( \leq 2n-1 \).

This problem cried out for MacMahon’s Partition Analysis, \ldots
“The no. of partitions of $N$ of the form $N = b_1 + \cdots + b_n$ satisfying

$$\begin{align*}
\frac{b_n}{n} &\geq \frac{b_{n-1}}{n-1} \geq \cdots \geq \frac{b_2}{2} \geq \frac{b_1}{1} \geq 0
\end{align*}$$

equals the no. of partitions of $N$ into odd parts each $\leq 2n - 1$.

This problem cried out for MacMahon’s Partition Analysis, . . .

Given that Partition Analysis is an algorithm for producing partition generating functions, I was able to convince Peter Paule and Axel Riese to join an effort to automate this algorithm.”
How Zeilberger tells the story of partition analysis (and more):
MacMahon’s Omega Calculus:

\[ T(x, y, z; q) := \sum_{c \geq b \geq a \geq 1, \text{s.t. } a+b>c} x^a y^b z^c q^{a+b+c} \]

\[ = \Omega \sum_{a,b,c \geq 1} l_1^{b-a} l_2^{c-b} l_3^{a+b-1-c} x^a y^b z^c q^{a+b+c} \]
MacMahon's Omega Calculus:

\[
T(x, y, z; q) := \sum_{c \geq b \geq a \geq 1, \text{ s.t. } a+b > c} \quad x^a y^b z^c q^{a+b+c} \\
= \Omega \sum_{a, b, c \geq 1} l_1^{b-a} l_2^{c-b} l_3^{a+b-1-c} \ x^a y^b z^c q^{a+b+c}
\]

\text{In[3]} := \text{OSum}[x^a y^b z^c q^{a+b+c}, \\
\{a \geq 1, b \geq a, c \geq b, a+b-1-c \geq 0\}, 1] \\
\text{Assuming } b \geq 0 \\
\text{Assuming } c \geq 0 \\
\text{Out[3]} = \Omega \sum_{l_1, l_2, l_3 \geq 1} \frac{q^x}{l_1 (1 - \frac{q z l_2}{l_3}) (1 - \frac{q y l_1 l_3}{l_2})}
\]
The second step is the elimination of the slack variables:

\[\text{Eliminating } l_2 \ldots\]
\[\text{Eliminating } l_3 \ldots\]
\[\text{Eliminating } l_1 \ldots\]

\[
\text{Out[4]} = \frac{q^3 x y z}{(1 - q^2 y z) (1 - q^3 x y z) (1 - q^4 x y z^2)}
\]
The second step is the elimination of the slack variables:

\[ \text{In[4]:=} \quad \text{OR [\%]} \]

Eliminating \( l_2 \ldots \)

Eliminating \( l_3 \ldots \)

Eliminating \( l_1 \ldots \)

\[ \text{Out[4]=} \quad \frac{q^3 x y z}{(1 - q^2 y z) (1 - q^3 x y z) (1 - q^4 x y z^2)} \]

NOTE. To this end, MacMahon used elimination rules like

\[ \Omega \frac{l^{-k}}{(1 - A l) (1 - \frac{B}{l})} = \frac{A^k}{(1 - A)(1 - AB)}, \quad k \geq 0. \]
GENERAL THEME: linear Diophantine constraints

Find $b_1, \ldots b_n \in \mathbb{N}$ such that

$$
\begin{pmatrix}
  c_{1,1} & \cdots & c_{1,n} \\
  c_{2,1} & \cdots & c_{2,n} \\
  \vdots & \ddots & \vdots \\
  c_{m,1} & \cdots & c_{m,n}
\end{pmatrix}
\begin{pmatrix}
  b_1 \\
  \vdots \\
  b_n
\end{pmatrix}
\geq
\begin{pmatrix}
  c_1 \\
  c_2 \\
  \vdots \\
  c_m
\end{pmatrix}
$$

New algorithm by F. Breuer & Z. Zafeirakopolous

GENERAL THEME: linear Diophantine constraints

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    \vdots & \ddots & \vdots \\
    c_{m,1} & \cdots & c_{m,n}
\end{pmatrix}
\begin{pmatrix}
    b_1 \\
    \vdots \\
    b_n
\end{pmatrix}
= 
\begin{pmatrix}
    c_1 \\
    c_2 \\
    \vdots \\
    c_m
\end{pmatrix}
$$

- New algorithm by F. Breuer & Z. Zafeirakopolous

Omega Discovery
MacMahon’s plane partitions, e.g., of $n = 3$:

\[
\begin{array}{ccccccc}
2 & \quad & 1 \\
3, 2 + 1, & + & 1 + 1 + 1, & 1, & + & 1 \\
1 & + & 1 & + & 1 \\
1 & & & & & & 1 \\
\end{array}
\]
MacMahon’s plane partitions, e.g., of $n = 3$:

\[
\begin{array}{cccc}
2 & 1 \\
3, 2 + 1, & 1 + 1 + 1, & 1, & 1 \\
1 & + & + \\
1 & & 1
\end{array}
\]

More generally, one can consider digraphs (resp. posets) like

\[
P = a_1 \quad a_2 \quad a_3 \quad a_4 \quad a_5 \quad a_6 \quad a_7 \quad a_8 \quad a_9 \quad a_{10}
\]
MacMahon’s plane partitions, e.g., of \( n = 3 \):

\[
\begin{array}{cccc}
2 & 1 \\
3, 2 + 1, & +, 1 + 1 + 1, & 1, & + \\
1 & + \\
1 \\
\end{array}
\]

More generally, one can consider digraphs (resp. posets) like

\[
P = a_1 \quad a_2 \quad a_3 \quad a_4 \quad a_5 \quad a_6 \quad a_7 \quad a_8 \quad a_9 \quad a_{10}
\]

together with the associated generating functions; here:

\[
f(q) := \sum_{a_1, \ldots, a_{10} \in \mathbb{N} \text{ s.t. } P} q^{a_1 + \cdots + a_{10}}.
\]
A $k$-elongated partition diamond of length 1

A $k$-elongated partition diamond of length $n$
Generating function for $k$-elongated diamonds of length $n$:

$$h_{n,k}(q) = \frac{\prod_{j=0}^{n-1} (1 + q^{(2k+1)j+2})(1 + q^{(2k+1)j+4}) \cdots (1 + q^{(2k+1)j+2k})}{\prod_{j=1}^{(2k+1)n+1} (1 - q^j)}$$

Andrews’ great idea: delete the source:
Generating function for $k$-elongated diamonds of length $n$:

$$h_{n,k}(q) = \frac{\prod_{j=0}^{n-1} (1 + q^{(2k+1)j+2})(1 + q^{(2k+1)j+4}) \cdots (1 + q^{(2k+1)j+2k})}{\prod_{j=1}^{(2k+1)n+1} (1 - q^j)}$$

Andrews’ great idea: delete the source:

$$h^*_{n,k}(q) = \frac{\prod_{j=0}^{n-1} (1 + q^{(2k+1)j+1})(1 + q^{(2k+1)j+3}) \cdots (1 + q^{(2k+1)j+2k-1})}{\prod_{j=1}^{(2k+1)n} (1 - q^j)}$$

and glue the diamonds together:
Generating function for $k$-elongated diamonds of length $n$:

$$h_{n,k}(q) = \frac{\prod_{j=0}^{n-1} (1 + q^{(2k+1)j+2}) (1 + q^{(2k+1)j+4}) \cdots (1 + q^{(2k+1)j+2k})}{\prod_{j=1}^{(2k+1)n+1} (1 - q^j)}$$

Andrews’ great idea: delete the source:

$$h_{n,k}^*(q) = \frac{\prod_{j=0}^{n-1} (1 + q^{(2k+1)j+1}) (1 + q^{(2k+1)j+3}) \cdots (1 + q^{(2k+1)j+2k-1})}{\prod_{j=1}^{(2k+1)n} (1 - q^j)}$$

and glue the diamonds together:

A broken $k$-diamond of length $2n$
Consequently,

\[
\sum_{m=0}^{\infty} \Delta_k(m) q^m := \lim_{n \to \infty} h_{n,k}(q) h_{n,k}^*(q)
\]

\[
= \prod_{j=1}^{\infty} \frac{(1 - q^{2j})(1 - q^{(2k+1)j})}{(1 - q^j)^3(1 - q^{(4k+2)j})}
\]

\[
= \frac{E(q^2)E(q^{2k+1})}{E(q)^3 E(q^{4k+2})}
\]

—and think of Ramanujan!
Consequently,

\[ \sum_{m=0}^{\infty} \Delta_k(m) q^m := \lim_{n \to \infty} h_{n,k}(q) h^*_{n,k}(q) \]

\[ = \prod_{j=1}^{\infty} \frac{(1 - q^{2j})(1 - q^{(2k+1)j})}{(1 - q^j)^3(1 - q^{(4k+2)j})} \]

\[ = \frac{E(q^2)E(q^{2k+1})}{E(q)^3E(q^{4k+2})} \]

— and think of Ramanujan!

A broken \( k \)-diamond of length \( 2n \)
In[1]:= \[ b \text{d}[n_, k_] := \prod_{j=1}^{n} \frac{(1-q^{2 j}) (1-q^{(2k+1) j})}{(1-q^j)^3 (1-q^{(4k+2) j})} \]

In[6]:= \[ b\text{d}1 = \text{Normal}[\text{Series}[\text{bd}[30, 1], \{q, 0, 30\}]] \]

\[ 1 + 3 q + 8 q^2 + 18 q^3 + 38 q^4 + 75 q^5 + 142 q^6 + 258 q^7 + 455 q^8 + 780 q^9 \]
\[ + 1308 q^{10} + 2148 q^{11} + 3467 q^{12} + 5505 q^{13} + 8618 q^{14} + 13314 q^{15} \]
\[ + 20327 q^{16} + 30693 q^{17} + 45882 q^{18} + 67944 q^{19} + 99745 q^{20} \]
\[ + 145239 q^{21} + 209882 q^{22} + 301128 q^{23} + 429148 q^{24} + 607710 q^{25} \]
\[ + 855414 q^{26} + 1197228 q^{27} + 1666585 q^{28} + 2308014 q^{29} + 3180668 q^{30} \]

In[7]:= \[ \text{Mod}[\text{CoefficientList}[\text{bd}1, q], 2] \]

\[ \{1, 1, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 1, 0, 1, 0, 0, 1, 0, 0, 0, 1, 0, \]
\[ 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0 \} \]

In[8]:= \[ \text{Mod}[\text{CoefficientList}[\text{bd}1, q], 3] \]

\[ \{1, 0, 2, 0, 2, 0, 1, 0, 2, 0, 0, 0, 0, 2, 0, 2, 0, 2, 0, 0, 0, 1, 0, \]
\[ 2, 0, 1, 0, 0, 0, 1, 0, 2 \} \]

In[9]:= \[ \text{Mod}[\text{CoefficientList}[\text{bd}1, q], 4] \]

\[ \{1, 3, 0, 2, 2, 2, 2, 3, 0, 0, 0, 0, 3, 1, 2, 2, 3, 1, 2, 2, 0, 1, 3, \]
Theorem. For all $n \in \mathbb{N}$,

$$\Delta_1(2n + 1) \equiv 0 \pmod{3}.$$
Theorem. For all $n \in \mathbb{N}$,

$$\Delta_1(2n + 1) \equiv 0 \pmod{3}.$$ 

Proof. Because of $(1 - q^j)^3 \equiv 1 - q^{3j} \pmod{3},$

$$
\sum_{m=0}^{\infty} \Delta_1(m) q^m = \prod_{j=1}^{\infty} \frac{(1 - q^{2j})(1 - q^{3j})}{(1 - q^j)^3(1 - q^{6j})}
\equiv \prod_{j=1}^{\infty} \frac{(1 - q^{2j})(1 - q^{3j})}{(1 - q^{3j})(1 - q^{6j})} \pmod{3}.
$$

Hence the coefficients of odd powers of $q$ have to be zero.
Recall:

**Theorem.** For all $n \in \mathbb{N}$,

$$\Delta_1(2n + 1) \equiv 0 \pmod{3}.$$ 

**Algorithmic Proof [Radu 2014]:**
Recall:

**Theorem.** For all $n \in \mathbb{N}$,\[
\Delta_1(2n + 1) \equiv 0 \pmod{3}.
\]

**Algorithmic Proof [Radu 2014]:**
\[
\sum_{n=0}^{\infty} \Delta_1(2n + 1)q^n = 3 \prod_{j=1}^{\infty} \frac{(1 - q^{2j})^2(1 - q^{6j})^2}{(1 - q^j)^6}
\]
Recall:

**Theorem.** For all $n \in \mathbb{N}$,

$$\Delta_1(2n + 1) \equiv 0 \pmod{3}.$$

**Algorithmic Proof [Radu 2014]:**

$$\sum_{n=0}^{\infty} \Delta_1(2n + 1)q^n = 3 \prod_{j=1}^{\infty} \frac{(1 - q^{2j})^2(1 - q^{6j})^2}{(1 - q^j)^6}$$

**NOTE.** Human proof [Hirschhorn & Sellers, 2007]
NOTE. There is much more: many new families of combinatorial objects, $q$-series identities, and computer proofs and findings.

Let us return to Ramanujan’s congruence

$$p(11n + 6) \equiv 0 \pmod{11} :$$
Radu’s “Ramanujan-Kolberg” package computes in $M^\infty(22)$:

$$\frac{E(q)^{10} E(q^2)^2 E(q^{11})^{11}}{E(q^{22})^{22}} \sum_{n=0}^{\infty} p(11n+6)q^n$$

$$= q^{14} (1078t^4 + 13893t^3 + 31647t^2 + 11209t - 21967$$

$$+ z_1 (187t^3 + 5390t^2 + 594t - 9581)$$

$$+ z_2 (11t^3 + 2761t^2 + 5368t - 6754))$$

with
Radu’s “Ramanujan-Kolberg” package computes in $M^\infty(22)$:

$$\frac{E(q)^{10}E(q^2)^2E(q^{11})^{11}}{E(q^{22})^{22}} \sum_{n=0}^\infty p(11n + 6)q^n = q^{14}(1078t^4 + 13893t^3 + 31647t^2 + 11209t - 21967$$

$$+ z_1(187t^3 + 5390t^2 + 594t - 9581)$$

$$+ z_2(11t^3 + 2761t^2 + 5368t - 6754))$$

with

$$t := \frac{3}{88}w_1 + \frac{1}{11}w_2 - \frac{1}{8}w_3, \quad z_1 := - \frac{5}{88}w_1 + \frac{2}{11}w_2 - \frac{1}{8}w_3 - 3,$$

$$z_2 := \frac{1}{44}w_1 - \frac{3}{11}w_2 + \frac{5}{4}w_3,$$

where
Radu’s “Ramanujan-Kolberg” package computes in $M^\infty(22)$:

$$
\frac{E(q)^{10}E(q^2)^2E(q^{11})^{11}}{E(q^{22})^{22}} \sum_{n=0}^{\infty} p(11n + 6)q^n
$$

$$
= q^{14}(1078t^4 + 13893t^3 + 31647t^2 + 11209t - 21967
+ z_1(187t^3 + 5390t^2 + 594t - 9581)
+ z_2(11t^3 + 2761t^2 + 5368t - 6754))
$$

with

$$
t := \frac{3}{88}w_1 + \frac{1}{11}w_2 - \frac{1}{8}w_3, \quad z_1 := -\frac{5}{88}w_1 + \frac{2}{11}w_2 - \frac{1}{8}w_3 - 3,
$$

$$
z_2 := \frac{1}{44}w_1 - \frac{3}{11}w_2 + \frac{5}{4}w_3,
$$

where

$$
w_1 := [-3, 3, -7], \quad w_2 := [8, 4, -8], \quad w_3 := [1, 11, -11] \in E^\infty(22)
$$

and
Radu’s “Ramanujan-Kolberg” package computes in $\mathcal{M}^\infty(22)$:

$$
\frac{E(q)^{10}E(q^2)^2E(q^{11})^{11}}{E(q^{22})^{22}} \sum_{n=0}^{\infty} p(11n+6)q^n
$$

$$
= q^{14}(1078t^4 + 13893t^3 + 31647t^2 + 11209t - 21967 + z_1(187t^3 + 5390t^2 + 594t - 9581) + z_2(11t^3 + 2761t^2 + 5368t - 6754))
$$

with

$$
t := \frac{3}{88}w_1 + \frac{1}{11}w_2 - \frac{1}{8}w_3, z_1 := -\frac{5}{88}w_1 + \frac{2}{11}w_2 - \frac{1}{8}w_3 - 3,
$$

$$
z_2 := \frac{1}{44}w_1 - \frac{3}{11}w_2 + \frac{5}{4}w_3,
$$

where

$$
w_1 := [-3, 3, -7], w_2 := [8, 4, -8], w_3 := [1, 11, -11] \in \mathcal{E}^\infty(22)
$$

and

$$
[a, b, c] := q^{-5} \frac{E(q^2)^a E(q^{11})^b E(q^{22})^c}{E(q)^{a+b+c}}.
$$
References


RISC Software & Appls.
Software of the RISC Algorithmic Combinatorics Group

Feel free to pick up packages!

www.risc.jku.at/research/combinat/software
Symbolic Summation

Hypergeometric Summation

- `fastZeil`, the Paule/Schorn implementation of Gosper's and Zeilberger's algorithm in Mathematica (by F. Paule).
- `Zeilberger`, a Maxima implementation of Gosper's and Zeilberger's algorithm (by F. Caruso).
- `MultiSum`, a Mathematica package for proving hypergeometric multi-sum identities (by K. Wegschaider).

$q$-Hypergeometric Summation

- `qZeil`, a Mathematica implementation of $q$-analogues of Gosper's and Zeilberger's algorithm (by A. Frieze).
- `Bibasic Telescope (pqTelescope)`, a Mathematica implementation of a generalization of Gosper's algorithm (by A. Riese).
- `qMultiSum`, a Mathematica package for proving $q$-hypergeometric multi-sum identities (by A. Riese).

Multi-Summation in Difference Fields

- `Sigma`, a Mathematica package for discovering and proving multi-sum identities (by C. Schneider).

Symbolic Summation for Stirling Numbers

- `Stirling`, a Mathematica package for computing recurrence equations of sums involving Stirling numbers.

Symbolic Summation and Integration for Holonomic Functions

- `HolonomicFunctions`, a Mathematica package for dealing with multivariate holonomic functions, including summation and integration.
Sequences and Power Series

- **Asymptotics**, a Mathematica package for computing asymptotic series expansions of univariate holonomic functions.
- **Dependencies**, a Mathematica package for computing algebraic relations of C-finite sequences and multi-indexed sequences.
- **Engel**, a Mathematica implementation of q-Engel Expansion (by B. Zimmermann).
- **GeneratingFunctions**, a Mathematica package for manipulating of univariate holonomic functions and sequences.
- **ore_algebra**, a Sage package for doing computations with Ore operators (by M. Kauers, M. Jaroschek, F. Jung).
- **qGeneratingFunctions**, a Mathematica package for manipulating of univariate q-holonomic functions and sequences.
- **Guess**, a Mathematica package for guessing multivariate recurrence equations (by M. Kauers).
- **RLangGFun**, a Maple implementation of the inverse Schützenberger methodology (by C. Koutschan).

Special Function Algorithms for Indefinite Nested Sums and Integrals

- **HarmonicSums**, a Mathematica package for dealing with harmonic sums, generalized harmonic sums and related objects.

Permutation Groups

- **PermGroup**, a Mathematica package for permutation groups, group actions and Polya theory (by T. Bayer, M. Kauers).
Partition Analysis

- **Omega**, a Mathematica implementation of Partition Analysis (by A. Riese).
- **GenOmega**, a Mathematica implementation of Guo-Niu Han's general Algorithm for MacMahon's Partition Analysis.

Difference/Differential Equations

- **DiffTools**, a Mathematica implementation of several algorithms for solving linear difference equations with polynomial coefficients.
- **OreSys**, a Mathematica implementation of several algorithms for uncoupling systems of linear Ore operator equations.
- **RatDiff**, a Mathematica implementation of Mark van Hoeij's algorithm for finding rational solutions of linear difference equations.
- **SumCracker**, a Mathematica implementation of several algorithms for identities and inequalities of special sequences.

Misc

- **Singular**, a Mathematica interface to the Singular system (by M. Kauers and V. Levandovskyy).
- **ModularGroup**, a Mathematica package providing basic algorithms and visualization routines related to the modular group (by T. Ponweiser).
Symbolic Summation in QFT

JKU Collaboration with DESY (Berlin–Zeuthen) (Deutsches Elektronen–Synchrotron)

Project leader: Carsten Schneider (RISC)
Partners: Johannes Blümlein (DESY)
Peter Paule (RISC)
Evaluation of Feynman diagrams

Behavior of particles

Evaluations required for the LHC experiment at CERN

Feynman integrals

DESY

simple sum expressions

processable by physicists

$\Phi(N, \epsilon, x) dx$

$\sum f(N, \epsilon, k)${multi-sums}

RISC

(symbolic summation)
Challenges of the project

About 1000 difficult Feynman diagrams have been treated so far

(some took 50 days of calculation time)

↓

About a million multi-sums have been simplified

(most were double and triple sums)

Resources

▶ 9 full time employed researchers at RISC/DESY:
  J. Ablinger, A. Behring, J. Blümlein, A. Hasselhuhn, A. de Freitas, C. Raab, M. Round, C. Schneider,
  F. Wissbrock

▶ 4 up-to-date mainframe DESY computers at RISC
  + exploiting DESY’s computer farms

▶ New computer algebra/special functions technologies
  (new/tuned algorithms, efficient implementations,...)
\[ = F_{-3}(N) \varepsilon^{-3} + F_{-2}(N) \varepsilon^{-2} + F_{-1}(N) \varepsilon^{-1} + \boxed{F_0(N)} \]

Simplify

\[
\sum_{j=0}^{N-3} \sum_{k=0}^{j} \sum_{l=0}^{k} \sum_{q=0}^{-j+N-3} \sum_{s=1}^{-l+N-q-3} \sum_{r=0}^{-l+N-q-s-3} (-1)^{-j+k-l+N-q-3} \times \\
\frac{(j+1)}{(k+1)(l+1)(j+2)} \frac{(N-1)}{q} \frac{(-j+N-3)}{s} \frac{(-l+N-q-3)}{r} \frac{(-l+N-q-s-3)}{r!(l+N-q-r-s-3)!} (s-1)! \\
\times \frac{(-l+N-q-2)!(-j+N-1)(N-q-r-s-2)(q+s+1)}{(-l+N-q-2)!(-j+N-1)(N-q-r-s-2)(q+s+1)} \\
\left[ 4S_1(-j+N-1) - 4S_1(-j+N-2) - 2S_1(k) \\
- (S_1(-l+N-q-2) + S_1(-l+N-q-r-s-3) - 2S_1(r+s)) \\
+ 2S_1(s-1) - 2S_1(r+s) \right] + 3 \text{ further 6–fold sums}
\]
\[
F_0(N) = \text{(using Sigma.m, EvaluateMultiSums.m and J. Ablinger's HarmonicSums.m package)}
\]
\[
\frac{7}{12} S_1(N)^4 + \frac{(17N + 5)S_1(N)^3}{3N(N + 1)} + \left(\frac{35N^2 - 2N - 5}{2N^2(N + 1)^2} + \frac{13S_2(N)}{2} + \frac{5(-1)^N}{2N^2}\right)S_1(N)^2
\]
\[
+ \left(-\frac{4(13N + 5)}{N^2(N + 1)^2} + \frac{4(-1)^N(2N + 1)}{N(N + 1)} - \frac{13}{N}\right)S_2(N) + \left(\frac{29}{3} - (-1)^N\right)S_3(N)
\]
\[
+ (2 + 2(-1)^N)S_{2,1}(N) - 28S_{-2,1}(N) + \frac{20(-1)^N}{N^2(N + 1)}S_1(N) + \left(\frac{3}{4} + (-1)^N\right)S_2(N)^2
\]
\[
- 2(-1)^N S_{-2}(N)^2 + S_{-3}(N)\left(\frac{2(3N - 5)}{N(N + 1)} + (26 + 4(-1)^N)S_1(N) + \frac{4(-1)^N}{N + 1}\right)
\]
\[
+ \left(\frac{(-1)^N(5 - 3N)}{2N^2(N + 1)} - \frac{5}{2N^2}\right)S_2(N) + S_{-2}(N)\left(10S_1(N)^2 + \frac{8(-1)^N(2N + 1)}{N(N + 1)}\right)
\]
\[
+ \frac{4(3N - 1)}{N(N + 1)}S_1(N) + \frac{8(-1)^N(3N + 1)}{N(N + 1)^2} + \left(-22 + 6(-1)^N\right)S_2(N) - \frac{16}{N(N + 1)}
\]
\[
+ \left(\frac{(-1)^N(9N + 5)}{N(N + 1)} - \frac{29}{3N}\right)S_3(N) + \left(\frac{19}{2} - 2(-1)^N\right)S_4(N) + \left(-6 + 5(-1)^N\right)S_{-4}(N)
\]
\[
+ \left(-\frac{2(-1)^N(9N + 5)}{N(N + 1)} - \frac{2}{N}\right)S_{2,1}(N) + (20 + 2(-1)^N)S_{2,-2}(N) + \left(-17 + 13(-1)^N\right)S_{3,1}
\]
\[
- \frac{8(-1)^N(2N + 1) + 4(9N + 1)}{N(N + 1)}S_{-2,1}(N) - (24 + 4(-1)^N)S_{-3,1}(N) + (3 - 5(-1)^N)S_{2,1}
\]
\[
+ 32S_{-2,1,1}(N) + \left(\frac{3}{2}S_1(N)^2 - \frac{3S_1(N)}{N} + \frac{3}{2}(-1)^N S_{-2}(N)\right)\zeta(2)
\]
\[
F_0(N) = \begin{align*}
&\frac{7}{12} S_1(N)^4 + \frac{(17N + 5) S_1(N)^3}{3N(N + 1)} + \left( \frac{35N^2 - 2N - 5}{2N^2(N + 1)^2} + \frac{13S_2(N)}{2} + \frac{5(-1)^N}{2N^2} \right) S_1(N)^2 \\
&+ \left( -\frac{4(13N + 5)}{N^2(N + 1)^2} + \frac{4(-1)^N(2N + 1)}{N(N + 1)} - \frac{13}{N} \right) S_2(N) + \left( \frac{29}{3} - (-1)^N \right) S_3(N) \\
&+ (2 + 2(-1)^N) S_{2,1}(N) - 28S_{-2,1}(N) + \frac{20(-1)^N}{N^2(N + 1)} S_1(N) + \left( \frac{3}{4} + (-1)^N \right) S_2(N)^2 \\
&- 2(-1)^N S_{-2,1,1}(N) - \sum_{i=1}^{N} \frac{1}{i^2} \sum_{j=1}^{N} \frac{1}{j} (-1)^N S_1(N) + \frac{4(-1)^N}{N^2(N + 1)} \\
&+ \left( \frac{3}{3N} S_3(N) + \left( -\frac{2}{2} + \frac{(-1)^N}{N} \right) S_4(N) + \left( -6 + 5(-1)^N \right) S_{-4}(N) \right) \\
&+ \left( -\frac{2(-1)^N(9N + 5)}{N(N + 1)} - \frac{2}{N} \right) S_{2,1}(N) + \left( 20 + 2(-1)^N \right) S_{2,-2}(N) + \left( -17 + 13(-1)^N \right) S_{3,1}(N) \\
&+ \left( -\frac{8(-1)^N(2N + 1) + 4(9N + 1)}{N(N + 1)} \right) S_{-2,1,1}(N) - \left( 24 + 4(-1)^N \right) S_{-3,1,1}(N) + \left( 3 - 5(-1)^N \right) S_{2,1,1}(N) \\
&+ 32S_{-2,1,1}(N) + \left( \frac{3}{2} S_1(N)^2 - \frac{3S_1(N)}{N} + \frac{3}{2} (-1)^N S_{-2}(N) \right) \zeta(2)
\end{align*}
\]
Symbolic Special Functions Manipulation: Application in Electromagnetic Wave Simulation

Collaboration with company CST (Computer Simulation Technology)

Partner: Joachim Schöberl (TU Vienna)
RISC: Christoph Koutschan (now RICAM)
Peter Paule
Simulation of electromagnetic waves

- joint work by Joachim Schöberl (RWTH Aachen), Peter Paule and Christoph Koutschan (RISC)
- wide range of applications in constructing antennas, mobile phones, etc.
- merchandised by the company CST (Computer Simulation Technology)

- simulation with finite element methods
- significant contributions from Symbolic Computation using CK’s package HolonomicFunctions
- symbolically derived formulae allow a considerable speed-up
- method is planned to be registered as a patent
Mathematical and physical background

Simulate the propagation of electromagnetic waves using the Maxwell equations

\[ \frac{dH}{dt} = \text{curl } E, \quad \frac{dE}{dt} = - \text{curl } H \]

where \( H \) and \( E \) are the magnetic and the electric field respectively.

Define basis functions (in 2D) in order to approximate the solution:

\[ \varphi_{i,j}(x, y) := (1 - x)^i P_j^{(2i + 1, 0)} (2x - 1) P_i \left( \frac{2y}{1 - x} - 1 \right) \]

Basis functions in 3D are more involved.
Results

In order to speed up the numerical computations, certain relations for the basis functions $\varphi_{i,j}(x, y)$ are needed.

Using HolonomicFunctions relations like the following can easily be derived:

\[
2(i + 2j + 5)(2i + 2j + 7) \frac{d}{dx} \varphi_{i,j+1}(x, y) \\
+ (2i + 1)(i + 2j + 1) \frac{d}{dx} \varphi_{i,j+2}(x, y) \\
- (j + 3)(i + 2j + 5) \frac{d}{dx} \varphi_{i,j+3}(x, y) \\
+ (j + 1)(2i + 2j + 7) \frac{d}{dx} \varphi_{i+1,j}(x, y) \\
- 2(2i + 3)(i + j + 3) \frac{d}{dx} \varphi_{i+1,j+1}(x, y) \\
+ (i + 2j + 5)(2i + 2j + 5) \frac{d}{dx} \varphi_{i+1,j+2}(x, y) = \\
2(i + j + 4)(2i + 2j + 5)(2i + 2j + 7) \varphi_{i,j+2}(x, y) \\
+ 2(i + j + 2)(i + 2j + 5)(2i + 2j + 7) \varphi_{i+1,j+1}(x, y)
\]

Much bigger formulae in the 3D case! Some efforts were needed to compute them.
Method

- basis functions \( \varphi_{i,j}(x, y) \) are composed of functions that are holonomic and \( \partial \)-finite, i.e., hypergeometric expressions and orthogonal polynomials (Legendre and Jacobi)
- differential equations and recurrence relations for these objects are known
- symbolic algorithms deliver relations for \( \varphi_{i,j}(x, y) \)
- compute a Gröbner basis for the ideal of such relations
- search in the ideal for relations of the desired form
- all the above steps can be performed automatically by HolonomicFunctions
The RISC Software Company: 
the applied branch of RISC

Industrial Applications

CEO: Wolfgang Freiseisen
50 employees
## About RISC Software GmbH

### RISC Institute
- Basic research in **Symbolic Computation**
- Chair: Peter Paule
- Founder (1987): Bruno Buchberger
- 50 employees (incl. PhD students)

### RISC Company
- **Software Development**
- Applied research in **Algorithmic Mathematics**
- Technology transfer
- 50 employees

### Ownership Structure
- 80% Johannes Kepler University
- 20% State Upper Austria (UAR GmbH)

### Business Units
- ISA
- Logistics Inf.
- Medical Inf.
- Advanced Computing Technologies

### Timeline

<table>
<thead>
<tr>
<th>Year</th>
<th>Event</th>
</tr>
</thead>
<tbody>
<tr>
<td>1989</td>
<td>Foundation of Softwarepark Hagenberg under the management of RISC</td>
</tr>
<tr>
<td>1990</td>
<td></td>
</tr>
<tr>
<td>1992</td>
<td>Foundation RISC Software GmbH (Prof. Bruno Buchberger)</td>
</tr>
<tr>
<td>1995</td>
<td>RISC SW specializes in software for logistics and production planning</td>
</tr>
<tr>
<td>2004</td>
<td>RISC Software GmbH becomes a 100%-subsidiary of JKU</td>
</tr>
<tr>
<td>2008</td>
<td>Incorporation of department Medical Informatics (State Upper Austria 20%)</td>
</tr>
<tr>
<td>2013</td>
<td>Headcount 52; annual turnover ca MEUR 4.5</td>
</tr>
</tbody>
</table>
Industrial Software Applications

- Simulation and Design
  - virtual product development
  - engineering workflow management
    Example: TRUMPF bending machines

- Manufacturing Processes and Control Systems
  - simulation of machining processes
    (e.g. NC programs)
  - geometric modeling and visualization
    Example: CrashGuard for WFL MillTurn
Software Development

- Computational Mathematics
  - Numerical and symbolic computation
  - Computational geometry and visualization

- Software Engineering
  - Algorithm engineering
  - Parallel computing
LAGRANGE – Multidisciplinary Structural Optimization Tool

- Software system 1984 (Airbus/EADS)
- since 2003: RISCSW coop.
- since 2009: main development partner of Airbus D&S for LAGRANGE

- RISCSW contributions
  - efficient, parallel algorithms
    (e.g. solution of linear systems with very special structures)
  - system architecture and development process
    (test management, etc.)
Appendix: Subalgebra Membership Algorithm
Application: $\mathbb{C}$-subalgebras of $\mathbb{C}[z]$

**GIVEN:** $f_1, f_2, f_3 \in \mathbb{C}[z]$ with

\[
\begin{align*}
  t := f_1 &= z^6 + a z^5 + \ldots, \\
  f_2 &= z^9 + b z^8 + \ldots, \\
  f_3 &= z^{20} + c z^{19} + \ldots;
\end{align*}
\]

**FIND:** $g_1, \ldots, g_k \in \mathbb{C}[z]$ such that

\[
\begin{align*}
  \mathbb{C}[f_1, f_2, f_3] &= \langle g_1, \ldots, g_k \rangle_{\mathbb{C}[t]} \\
  &= \mathbb{C}[t] + \mathbb{C}[t] g_1 + \cdots + \mathbb{C}[t] g_k.
\end{align*}
\]
Application: \( \mathbb{C}\)-subalgebras of \( \mathbb{C}[z] \)

**GIVEN:** \( f_1, f_2, f_3 \in \mathbb{C}[z] \) with

\[
    t := f_1 = z^6 + a z^5 + \ldots,
    \]
\[
        f_2 = z^9 + b z^8 + \ldots,
        \]
\[
            f_3 = z^{20} + c z^{19} + \ldots;
            \]

**FIND:** \( g_1, \ldots, g_k \in \mathbb{C}[z] \) such that

\[
    \mathbb{C}[f_1, f_2, f_3] = \langle g_1, \ldots, g_k \rangle_{\mathbb{C}[t]}
    = \mathbb{C}[t] + \mathbb{C}[t] g_1 + \cdots + \mathbb{C}[t] g_k.
\]

**KEY IDEA:**

\[
    z^n + \cdots = z^{u_i + \ell \odot 6} + \cdots = (z^{a_i \odot 9 + b_i \odot 20} + \cdots) t^\ell + \cdots.
\]
KEY PROCEDURE

GIVEN: $f_1, f_2, f_3 \in \mathbb{C}[z]$ with

$$M = \langle \deg(f_1), \deg(f_2), \deg(f_3) \rangle = \langle 6, 9, 20 \rangle;$$

select $t := f_1$ and define $m := \deg(t) = 6$;
KEY PROCEDURE

GIVEN: \( f_1, f_2, f_3 \in \mathbb{C}[z] \) with
\[
M = \langle \deg(f_1), \deg(f_2), \deg(f_3) \rangle = \langle 6, 9, 20 \rangle;
\]
select \( t := f_1 \) and define \( m := \deg(t) = 6 \);

FIND: \( g_1, \ldots, g_5 \in \mathbb{C}[f_1, f_2, f_3] \) such that
\[
\mathbb{C}[f_1, f_2, f_3] = \mathbb{C}[t, f_2, f_3] = \mathbb{C}[t, g_1, \ldots, g_5]
\]
and
\[
\{\deg(g_1), \ldots, \deg(g_5)\} = \{u_1, \ldots, u_5\} \equiv \{1, \ldots, 5\} \pmod{6}.
\]

How to determine the \( g_i \)?
How to determine the $g_i$?

$[M]_0 = \{0, 6, 12, \ldots \}$, and $t := f_1 = z^6 + a z^5 + \cdots$.

$[M]_1 = \{1, 7, 13, 19, 25, 31, 37, 43, 49, 54, \ldots \}$, define in view of $u_1 = 49 = \boxed{9} + 2 \boxed{20}$,

\[ g_1 := f_2 f_3^2 = z^{u_1} + \cdots. \]

$[M]_2 = \{2, 8, 14, 20, 26, \ldots \}$, define in view of $u_2 = \boxed{20}$,

\[ g_2 := f_3 = z^{u_2} + \cdots. \]

$[M]_3 = \{3, 9, 15, \ldots \}$, define in view of $u_3 = \boxed{9}$,

\[ g_3 := f_2 = z^{u_3} + \cdots. \]
$[M]_4 = \{4, 10, 16, 22, 28, 34, 40, 46, \ldots \}$, define in view of 
$u_4 = 40 = 2 \cdot 20$, 
\[ g_4 := f_3^2 = z^{u_4} + \cdots. \]

$[M]_5 = \{5, 11, 17, 23, 29, 35, \ldots \}$, define in view of 
$u_5 = 29 = 9 + 20$, 
\[ g_5 := f_2 f_3 = z^{u_5} + \cdots. \]
Combinatorics, Modular Functions, and Computer Algebra / Appendix: Subalgebra Membership Algorithm

\([M]_4 = \{4, 10, 16, 22, 28, 34, 40, 46, \ldots \}\), define in view of

\[ u_4 = 40 = 2 \times 20, \]

\[ g_4 := f_3^2 = z^{u_4} + \cdots. \]

\([M]_5 = \{5, 11, 17, 23, 29, 35, \ldots \}\), define in view of

\[ u_5 = 29 = 9 + 20, \]

\[ g_5 := f_2 f_3 = z^{u_5} + \cdots. \]

---

**SUMMARY.** We determined the \( g_i \in \mathbb{C}[f_1, f_2, f_3] \) as

\[
\begin{align*}
g_1 & := f_2 f_3^2 = z^{u_1} + \cdots, \\
g_2 & := f_3 = z^{u_2} + \cdots, \\
g_3 & := f_2 = z^{u_3} + \cdots, \\
g_4 & := f_3^2 = z^{u_4} + \cdots, \\
g_5 & := f_2 f_3 = z^{u_5} + \cdots,
\end{align*}
\]

such that

\[ \mathbb{C}[f_1, f_2, f_3] = \mathbb{C}[t, g_1, \ldots, g_5] \text{ and } \text{deg}(g_i) \equiv i \pmod{6}. \]
Example. \( f = z^{62} + \cdots \in \mathbb{C}[f_1, f_2, f_3] : \)

\[
62 = \boxed{20} + 7 \boxed{6} = u_2 + 7 \boxed{6};
\]
recalling \( g_2 := f_3 = z^{u_2} + \cdots \) one has,

\[
f - g_2 t^7 = c_1 z^{61} + \cdots .
\]

---

\[
61 = \boxed{49} + 2 \boxed{6} = u_1 + 2 \boxed{6};
\]
recalling \( g_1 := f_2 f_3^2 = z^{u_1} + \cdots \) one has,

\[
f - g_2 t^7 - c_1 g_1 t^2 = c_2 z^{60} + \cdots .
\]

BUT:
Example. \( f = z^{62} + \cdots \in \mathbb{C}[f_1, f_2, f_3] \):

\[
62 = 20 + 7 \cdot 6 = u_2 + 7 \cdot 6;
\]

recalling \( g_2 := f_3 = z^{u_2} + \cdots \) one has,

\[
f - g_2 t^7 = c_1 z^{61} + \cdots .
\]

\[
61 = 49 + 2 \cdot 6 = u_1 + 2 \cdot 6;
\]

recalling \( g_1 := f_2 f_3^2 = z^{u_1} + \cdots \) one has,

\[
f - g_2 t^7 - c_1 g_1 t^2 = c_2 z^{60} + \cdots .
\]

**BUT:** For instance, it could be that

\[
f - g_2 t^7 - c_1 g_1 t^2 = z^4 + \cdots \in \mathbb{C}[f_1, f_2, f_3].
\]
Example (contd). $f = z^{62} + \cdots \in \mathbb{C}[f_1, f_2, f_3]$:

For instance, it could be that

$$f - g_2 t^7 - c_1 g_1 t^2 = z^4 + \cdots \in \mathbb{C}[f_1, f_2, f_3].$$

NOTE.

$4 \notin \langle 6, 9, 20 \rangle = \langle \deg(f_1), \deg(f_2), \deg(f_3) \rangle$

BUT

$4 \in \langle \deg(g) : g \in \mathbb{C}[f_1, f_2, f_3] \rangle.$

SOLUTION.
Example (contd). \( f = z^{62} + \cdots \in \mathbb{C}[f_1, f_2, f_3] \): For instance, it could be that

\[
f - g_2 t^7 - c_1 g_1 t^2 = z^4 + \cdots \in \mathbb{C}[f_1, f_2, f_3].
\]

NOTE. \( 4 \notin \langle 6, 9, 20 \rangle = \langle \deg(f_1), \deg(f_2), \deg(f_3) \rangle \)

BUT \( 4 \in \langle \deg(g) : g \in \mathbb{C}[f_1, f_2, f_3] \rangle \).

SOLUTION. We repeat the KEY PROCEDURE

\[\{f_1, f_2, f_3\} \rightsquigarrow \{g_1, g_2, g_3, g_4, g_5\}\]

by iteratively joining products:

\[\{g_1, g_2, g_3, g_4, g_5\} \cup \{g_i g_j\} \rightsquigarrow \{h_1, h_2, h_3, h_4, h_5\},\]

\[\{h_1, h_2, h_3, h_4, h_5\} \cup \{h_i h_j\} \rightsquigarrow \{\ldots\}, \text{ a.s.o.}\]
DECISION PROCEDURE: \( \mathbb{C} \)-subalgebra membership

GIVEN: \( f \) and \( f_1, \ldots, f_n \) from \( \mathbb{C}[z] \);

FIND: polynomials \( p_0, p_1, \ldots, p_k \in \mathbb{C}[z] \) such that

\[
    f = p_0(t) + p_1(t) \ g_1 + \cdots + p_k(t) \ g_k
\]

\( \in \langle 1, g_1, \ldots, g_k \rangle_{\mathbb{C}[t]} = \mathbb{C}[f_1, \ldots, f_n] \)

APPLICATION.
DECISION PROCEDURE: \( \mathbb{C} \)-subalgebra membership

**GIVEN:** \( f \) and \( f_1, \ldots, f_n \) from \( \mathbb{C}[z] \);

**FIND:** polynomials \( p_0, p_1, \ldots, p_k \in \mathbb{C}[z] \) such that

\[
 f = p_0(t) + p_1(t) g_1 + \cdots + p_k(t) g_k \\
 \in \langle 1, g_1, \ldots, g_k \rangle_{\mathbb{C}[t]} = \mathbb{C}[f_1, \ldots, f_n]
\]

**APPLICATION.** With this procedure the right hand sides of Ramanujan identities can be found automatically: just apply it to given \( f \in M^\infty(N) \) and \( f_1, \ldots, f_n \in E^\infty(N) \). In this context, \( z = \frac{1}{q} \) because of possible poles sitting at \([\infty]\); the \( f_j \) can be treated as formal Laurent series like

\[
 f_1 = \frac{a_6}{q^6} + \frac{a_5}{q^5} + \cdots , \\
 f_2 = \frac{b_9}{q^9} + \frac{b_8}{q^8} + \cdots , \\
 f_3 = \frac{c_{20}}{q^{20}} + \frac{c_{19}}{q^{19}} + \cdots , \text{ etc.}
\]
Example. Radu’s “Ramanujan-Kolberg” package in STEP 1 computes that:

\[ q^{-1} \frac{E(q)^8}{E(q^7)^7} \sum_{n=0}^{\infty} p(7n + 5)q^n \in M^\infty(7). \]
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In STEP 2 Radu’s package determines that the monoid is generated by one element (recall \( f_1 = t \)):

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In STEP 3 the package determines the module presentation

$$\mathbb{C}[f_1, \ldots, f_n] = \mathbb{C}[f_1] = \langle 1, g_1, \ldots, g_k \rangle_{\mathbb{C}[t]} = \langle 1 \rangle_{\mathbb{C}[f_1]}.$$
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In STEP 4 the package finds Ramanujan's identity:

\[ q^{-1} \frac{E(q)^8}{E(q^7)^7} \sum_{n=0}^{\infty} p(7n + 5)q^n = 49 \cdot 1 + 7 \cdot f_1. \]