Parity-check codes and their representations
Gretchen L. Matthews
July 21, 2015

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Linear codes can be represented by parity-check matrices.

Let $F_q$ be the finite field with $q$ elements, where $q$ is a power of a prime. A linear code $C$ of length $n$ over $F_q$ is a subspace of $F_q^n$. Elements of $C$ are called codewords.
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$$C = NS(H).$$
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$$C = \text{NS}(H).$$

The matrix $H$ is called a parity-check matrix for $C$, because

$$y \in C \text{ if and only if } Hy^T = 0.$$ 

The code $C$ is sometimes called a parity-check code. If $H$ is sparse, then $C$ is a low-density parity-check (LDPC) code.
A parity-check matrix for a linear code is not unique.

Example

Consider the binary code

\[ C = \{ (0,0,0,0,0,0,0), (0,0,1,0,1,1,1), (0,1,0,1,0,1,1), (0,1,1,1,1,0,0), (1,0,0,1,1,0,1), (1,0,1,1,0,1,0), (1,1,0,0,1,1,0), (1,1,1,0,0,0,1) \} \subseteq \mathbb{F}_2. \]
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Then \( C \) has

\[
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 1 & 0 & 0 & 1 & 0 & 1 \\
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\end{pmatrix}
\]

as a parity-check matrix.
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\end{pmatrix}
\]

as a parity-check matrix. Another parity-check matrix for \( C \) is

\[
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 1 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 & 1 & 1 & 0 \\
0 & 0 & 1 & 0 & 1 & 1 & 1
\end{pmatrix}.
\]
A code can be represented by a Tanner graph.

The Tanner graph of a binary matrix \( H \) is a bipartite graph \( T(H) = (X \cup F, E) \) with biadjacency matrix \( H \), meaning

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$$H = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$
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- $\{x_i, f_j\} \in E$ if and only if $h_{ji} = 1$.

Example

$$H = \begin{pmatrix}
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Tanner graph of $H$:
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**Example**

Tanner graph of $H$:
Tanner graphs of \( q \)-ary codes are weighted bipartite graphs.

The Tanner graph \( T(H) \) of \( H \in \mathbb{F}_q^{r \times n} \) is a weighted bipartite graph with biadjacency matrix \( H \); that is,

- the vertex set of \( T(H) \) is \( \{x_1, \ldots, x_n\} \cup \{f_1, \ldots, f_r\} \),
- the edge set of \( T(H) \) is \( \{\{x_i, f_j\} : h_{ji} \neq 0\} \), and
- edge weights are given by \( \text{wt} \{x_i, f_j\} = h_{ji} \).

Example

The Tanner graph of \( H = \begin{bmatrix} 1 & 2 & 2 & 1 \\ 2 & 0 & 1 & 2 \end{bmatrix} \) over \( \mathbb{F}_3 \) is

\[
\begin{array}{c}
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A code may have more than one Tanner graph.

The Tanner graph of a code depends on the choice of parity-check matrix.

**Example**

Tanner graphs $T(H_1)$ and $T(H_2)$ for the same binary code are shown below.

\[
H_1 = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 1 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 & 1 & 1 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 1
\end{pmatrix} \quad H_2 = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 1 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 & 1 & 1 & 0 \\
0 & 0 & 1 & 0 & 1 & 1 & 1
\end{pmatrix}
\]

To emphasize that a code $C$ is being considered with a fixed parity-check matrix $H$, we write

\[C = C(H)\]
Codewords may be represented graphically.

An element $c = (c_1, c_2, \ldots, c_n)$ is a codeword of a binary code $C = C(H)$ if and only if the binary assignment $(c_1, c_2, \ldots, c_n)$ to the bit nodes of $T(H)$ make the binary sum at every check node 0.

**Example**

$$ (1, 1, 1, 0, 0, 0, 0, 0) $$

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Thus, $(1, 1, 1, 0, 0, 0, 0)$ is a codeword.
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Example

(1, 1, 1, 1, 0, 0, 0) corresponds to

Thus, (1, 1, 1, 0, 0, 0, 0) is a codeword. However, (1, 1, 1, 1, 0, 0, 0) is not a codeword.
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Example

The Tanner graph of $H = \begin{bmatrix} 1 & 2 & 2 & 1 \\ 2 & 0 & 1 & 2 \end{bmatrix}$ over $\mathbb{F}_3$ is

Note that $(1, 0, 2, 1)$ is a codeword of the code defined by $H$. 

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- the recursive construction of codes and
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- providing bounds on code parameters.

The Hamming distance between two words \(w, y \in \mathbb{F}_q^n\) is

\[
d(w, y) := \# \{i : w_i \neq y_i\}.
\]

Code parameters

Classically, the quality of a code \(C\) is measured by parameters
- \(n\), the length of the code \(C\)
- \(k := \dim_{\mathbb{F}_q} C\), the dimension of the code \(C\)
- \(d := \min \{d(c, c') : c, c' \in C, c \neq c'\}\), the minimum distance of \(C\)

A code \(C\) with these parameters is called an \([n, k, d]\) code.
Given a received word, a decoder seeks to find the most likely codeword sent.

**Minimum distance decoding problem**

Given a received word $w \in \mathbb{F}_q^n$, find $y \in C$ such that

$$d(w, y) \leq d(w, c) \quad \forall c \in C.$$
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P(y \text{ sent} | w \text{ received}),
\]

the probability that the codeword \( y \) was sent given that \( w \) is received.

Maximum likelihood decoding is equivalent to minimum distance decoding (provided the channel has error probability \(< .5\)).
The ML decoding problem can be stated as an LP.

**Definition**

Given a binary code $C$ of length $n$,

$$poly(C) := \left\{ \sum_{y \in C} \lambda_y y : \lambda_y \geq 0, \sum_{y \in C} \lambda_y = 1 \right\} \subseteq [0, 1]^n$$

denotes the codeword polytope of $C$. 

**Definition**

Let $i := \log \frac{P(w_i | y_i = 0)}{P(w_i | y_i = 1)}$ denote the negative log-likelihood ratio at the $i$th coordinate.
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Let

$$\gamma_i := \log \left( \frac{P(w_i | y_i = 0)}{P(w_i | y_i = 1)} \right)$$

denote the negative log-likelihood ratio at the $i^{th}$ coordinate.

The ML decoding problem is equivalent to the following linear program (LP):

$$\text{minimize } \sum_{i=1}^{n} \gamma_i f_i \text{ subject to } f \in poly(C).$$
The ML decoding problem can be stated as an LP.

Write $\mathbb{F}_q = \{0, \alpha, \ldots, \alpha^{q-1}\}$, and define $\phi : \mathbb{F}_q^n \rightarrow \{0, 1\}^{(q-1)n}$ $x \mapsto (e_{x_1}, e_{x_2}, \ldots, e_{x_n})$.

**Definition**

Given a code $C$ of length $n$ over $\mathbb{F}_q$, the codeword polytope of $C$ is

$$\text{poly}(C) := \left\{ \sum_{c \in C} \lambda_c \phi(c) : \lambda_c \geq 0, \sum_{c \in C} \lambda_c = 1 \right\} \subseteq [0, 1]^{(q-1)n}.$$
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\]

The ML decoding problem for \( C \) is equivalent to the following LP:

\[
\text{minimize} \quad \sum_{k=1}^{(q-1)n} \gamma_k f_k \quad \text{subject to} \quad f \in \text{poly}(C),
\]

where \( \gamma_{(q-1)(i-1)+j} := \log \left( \frac{P(y_i|0)}{P(y_i|\alpha^j)} \right) \) is a log-likelihood ratio at the \( i^{th} \) coordinate and \( j = 1, \ldots, q - 1 \).
LP relaxation of ML decoding is more manageable.

**Definition**

Suppose $C = C(H)$ be a code over $\mathbb{F}_q$ of length $n$ and $Row_j(H)$ denote the $j^{th}$ row of $H$. The fundamental polytope of $H$ is

$$Q(H) = \cap_{j=1}^r \text{poly} \left( C \left( Row_j(H) \right) \right) \subseteq [0, 1]^{(q-1)n}.$$
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**Example**

Consider again

$$H_1 = \begin{pmatrix}
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$$Q(H) = \bigcap_{j=1}^{r} poly(C(Row_j(H))) \subseteq [0, 1]^{(q-1)n}.$$ 

Note: $poly(C) \subseteq Q(H)$. 

[Linear Code Linear Program] [Feldman, Wainwright, & Karger, 2005; Flanagan, Skachek, Byrne, & Greferath, 2009]
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$$Q(H) = \cap_{j=1}^r \text{poly} (C(\text{Row}_j(H))) \subseteq [0, 1]^{(q-1)n}.$$

Note: $\text{poly} (C) \subseteq Q(H)$.

**Linear Code Linear Program** [Feldman, Wainwright, & Karger, 2005; Flanagan, Skachek, Byrne, & Greferath, 2009]

The Linear Code LP is

$$\text{minimize } \sum_{i=1}^{(q-1)n} \gamma_i f_i \text{ subject to } f \in Q(H).$$
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$$\text{minimize } \sum_{i=1}^{(q-1)n} \gamma_i f_i \text{ subject to } f \in Q(H).$$

**Definition**

The vertices of $Q(H)$ are called pseudocodewords.
LP relaxation may lead to a noncodeword output, called a pseudocodeword.

Some, but not necessarily all, pseudocodewords are codewords.
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If $T(H)$ is acyclic, then the Linear Code LP and ML decoding for $C(H)$ are equivalent.
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If $T(H)$ is acyclic, then the Linear Code LP and ML decoding for $C(H)$ are equivalent.

**Theorem [Etzion, Trachtenberg, & Vardy, 1999]**

Given an $[n, k, d]$ code with acyclic Tanner graph, either $\frac{k}{n} \leq 0.5$ or

$$\frac{k}{n} > 0.5 \text{ and } d = 2.$$
Pseudocodewords can also be defined via graph covers.

Example

Consider the Tanner graph $T(H)$:

![Tanner graph](image)
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**Example**

Consider the Tanner graph $T(H)$:

A degree 2 cover of $T(H)$ is
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The degree 2 cover gives rise to codewords:

In this case, we obtain $\frac{1}{2}(1, 1, 1, 2, 1, 1, 1)$. 
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$$\frac{1}{M}(m_1(1), \ldots, m_n(1)) \in [0, 1]^n$$

where $m_i(b)$ is the number of copies of the $i^{th}$ symbol that take the value $b \in \mathbb{F}_2$ in a codeword $c$ from $C(\tilde{T}(H))$

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Let $C = C(H)$ be a code over $\mathbb{F}_q$ of length $n$. Pseudocodewords from a degree $M$ graph cover $\tilde{T}(H)$ of $T(H)$ are of the form

$$\frac{1}{M}(m_1(\alpha), \ldots, m_n(\alpha), \ldots, m_1(\alpha^{q-1}), \ldots, m_n(\alpha^{q-1})) \in [0, 1]^{(q-1)n}$$

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Example

The ternary code $C(H)$ with $H = \begin{bmatrix} 1 & 2 & 2 & 1 \\ 2 & 0 & 1 & 2 \end{bmatrix} \in \mathbb{F}_3^{2\times 4}$
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![Graph Cover Diagram]
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$$\tilde{c} = (2, 0, 2, 1, 0, 1, 1, 1, 2, 1, 1, 0, 0, 2, 0, 0)$$
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$$\tilde{c} = (2, 0, 2, 1, 0, 1, 1, 1, 2, 1, 1, 0, 0, 2, 0, 0)$$

that gives rise to the pseudocodeword \( \left( \frac{1}{4}, \frac{3}{4}, \frac{2}{2}, 0, \frac{2}{2}, 0, \frac{1}{4}, \frac{1}{4} \right) \).
LP decoding gives an approximation to iterative decoding algorithms.

Iterative decoders make use of local computations, while ML decoders search for a configuration that satisfies all check nodes.

Advantages of an iterative decoder:

- Can correct more errors than guaranteed by ML decoding
- Does not require any particular structure
- Fast (linear in code length for sparse matrices) and easy to implement
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Aim
To provide tools for determining the expected performance of a parity-check code and for selecting parity-check matrices which lessen the impact of noncodeword pseudocodewords.
Two tools for studying pseudocodewords: generating functions and the fundamental cone.

**Notation**

Given a matrix $H$, let $\mathcal{P}(H)$ denote the set of pseudocodewords of $C(H)$

**Generating function of the pseudocodewords**

Given a code $C = C(H)$, we consider

$$\sum_{p \in \mathcal{P}(H)} x^p.$$
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$$\sum_{p \in \mathcal{P}(H)} x^p.$$ 

Fundamental cone
The fundamental cone $\mathcal{K}(H)$ of a code $C(H)$ of length $n$ over $\mathbb{F}_q$ is the smallest cone in $\mathbb{R}^{(q-1)n}$ that contains all pseudocodewords of $C(H)$. 

Gretchen L. Matthews
Two tools for studying pseudocodewords: generating functions and the fundamental cone.

**Theorem (Koetter, Li, Vontobel, & Walker 2007)**

Let $H \in \mathbb{F}_2^{r \times n}$. Given $p \in \mathbb{Z}^n$, the following are equivalent:

1. $p$ is a pseudocodeword of $C(H)$ and
2. $p \in \mathcal{K}(H)$ and $Hp^T = 0 \in \mathbb{F}_2^r$.

**Theorem (Koetter, Li, Vontobel, & Walker 2007)**

Suppose $H \in \mathbb{F}_2^{r \times n}$ is a matrix with exactly two 1’s in each column. Then the generating function of the pseudocodewords of $C(H)$ is

$$\sum_{p \in \mathcal{P}(H)} x^p = \zeta_{\mathcal{N}(H)}(x),$$

the edge zeta function of the normal graph of $T(H)$.
The generating function for the pseudocodewords of a code is a rational function.

**Theorem (K-M)**

Given a binary matrix $H \in \mathbb{F}_2^{r \times n}$,

- the generating function for the pseudocodewords of $C(H)$, $\sum_{p \in \mathcal{P}(H)} x^p$, is a rational function; and
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**Theorem (K-M)**

Given a binary matrix $H \in \mathbb{F}_2^{r \times n}$,
- the generating function for the pseudocodewords of $C(H)$, $\sum_{p \in \mathcal{P}(H)} x^p$, is a rational function; and
- there exists a polynomial time algorithm which computes the generating function of the pseudocodewords, $\sum_{p \in \mathcal{P}(H)} x^p$, as a finite sum

$$\sum_{i} \alpha_i \frac{x^{u_i}}{(1 - x^{w_{i1}}) \ldots (1 - x^{w_{id}})}$$

where $\alpha_i \in \{0, 1\}$ and $u_i, w_{ij}$ are integer vectors for all $i, j$. 
Barvinok’s algorithm enumerates integer points of a rational cone.

Barvinok’s algorithm takes as input a pointed rational cone described in terms of rational inequalities and produces a generating function that enumerates its integer points.
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Recall that the fundamental cone of a binary code $C(H)$ is

$$\mathcal{K}(H) := \{ v \in \mathbb{R}^n : Row_j H v^T - 2h_{ji} v_i \geq 0, v_i \geq 0 \ \forall 1 \leq i \leq n, 1 \leq j \leq r \}.$$
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$$K(H) := \{v \in \mathbb{R}^n : Row_j Hv^T - 2h_{ji}v_i \geq 0, v_i \geq 0 \ \forall 1 \leq i \leq n, 1 \leq j \leq r\}.$$ 

However, the pseudocodewords of $C(H)$ are those integer points of $K(H)$ which satisfy $Hp^T = \mathbf{0} \in \mathbb{F}_2^r$. 

Gretchen L. Matthews
Lifting the fundamental cone allows us to apply Barvinok’s algorithm.

Given $H \in \mathbb{F}_2^{r \times n}$, the fundamental cone of $C(H)$ is

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**Definition**

Given $H \in \mathbb{F}_2^{r \times n}$, the lifted fundamental cone of $C(H)$ is

$$\hat{\mathcal{K}}(H) = \left\{ (v, a) \in \mathbb{R}^{n+r} \mid v_i \geq 0, Hv^T = 2a^T, \text{ and} \begin{align*} \text{Row}_j Hv^T - 2h_{ji}v_i & \geq 0 \\
\text{for all} & \ 1 \leq i \leq n \ \text{and} \ 1 \leq j \leq r \end{align*} \right\}.$$
Lifting the fundamental cone allows us to apply Barvinok’s algorithm.

Given $H \in F_{2}^{r \times n}$, the fundamental cone of $\mathcal{C}(H)$ is

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**Definition**

Given $H \in F_{2}^{r \times n}$, the lifted fundamental cone of $\mathcal{C}(H)$ is

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Consider the projection

$$\pi : \mathbb{R}^{n+r} \rightarrow \mathbb{R}^{n} \quad (v, a) \mapsto v.$$
Lifting the fundamental cone allows us to apply Barvinok’s algorithm.

Given \( H \in \mathbb{F}_2^{r \times n} \), the fundamental cone of \( C (H) \) is

\[
\mathcal{K} (H) := \left\{ v \in \mathbb{R}^n : \text{Row}_j Hv^T - 2h_{ji} v_i \geq 0, v_i \geq 0 \ \forall 1 \leq i \leq n, 1 \leq j \leq r \right\}.
\]

**Definition**

Given \( H \in \mathbb{F}_2^{r \times n} \), the lifted fundamental cone of \( C (H) \) is

\[
\hat{\mathcal{K}} (H) = \left\{ (v, a) \in \mathbb{R}^{n+r} : \begin{array}{c}
v_i \geq 0, Hv^T = 2a^T, \\
\text{Row}_j Hv^T - 2h_{ji} v_i \geq 0 \\
\text{for all } 1 \leq i \leq n \text{ and } 1 \leq j \leq r
\end{array} \right\}.
\]

Consider the projection

\[
\pi : \mathbb{R}^{n+r} \rightarrow \mathbb{R}^n
\]

\[
(\mathbf{v}, \mathbf{a}) \mapsto \mathbf{v}.
\]

**Proposition**

Let \( H \in \mathbb{F}_2^{r \times n} \). Then \( \pi \left( \hat{\mathcal{K}} (H) \right) = \mathcal{K} (H) \) and \( \mathcal{P} (H) = \pi \left( \hat{\mathcal{K}} (H) \cap \mathbb{Z}^{n+r} \right) \).
The generating function for the pseudocodewords of a code is a rational function.

Denote the generating function for integer points in the lifted fundamental cone by

\[ f(x_1, x_2, \ldots, x_{n+r}) := \sum_{(v,a) \in \hat{C}(H) \cap \mathbb{Z}^{n+r}} x^{(v,a)}. \]
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Then \( f(x_1, x_2, \ldots, x_{n+r}) \) can be expressed as a rational function. Applying the Barvinok-Woods approach to specialization gives that

\[ \sum_{v \in \mathcal{P}(H)} x^v = \sum_{v \in \pi(\hat{K}(H) \cap \mathbb{Z}^{n+r})} x^v = \sum_{(v, a) \in \hat{K}(H) \cap \mathbb{Z}^{n+r}} x^v = f(x_1, x_2, \ldots, x_n, 1, 1, \ldots, 1) \]

is rational.
The generating function for the pseudocodewords of a code is a rational function.

Denote the generating function for integer points in the lifted fundamental cone by

$$f(x_1, x_2, \ldots, x_{n+r}) := \sum_{(v,a) \in \hat{\mathcal{C}}(H) \cap \mathbb{Z}^{n+r}} x^{(v,a)}.$$ 

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$$\sum_{v \in \mathcal{P}(H)} x^v = \sum_{v \in \pi(\hat{\mathcal{C}}(H) \cap \mathbb{Z}^{n+r})} x^v = \sum_{(v,a) \in \hat{\mathcal{C}}(H) \cap \mathbb{Z}^{n+r}} x^v = f(x_1, x_2, \ldots, x_n, 1, 1, \ldots, 1)$$

is rational.

**Theorem (K-M)**

*Given $H \in \mathbb{F}_2^{r \times n}$, the generating function for the pseudocodewords of $C(H)$ $\sum_{p \in \mathcal{P}(H)} x^p$ is a rational function.*
Critical multisets play a key role in understanding pseudocodewords.

**Definition**

Given a prime $p$, a critical multiset of $\mathbb{F}_p$ is a multiset \{\(\gamma_1, \ldots, \gamma_t\} \subseteq \{0, 1, \ldots, p - 1\}$ with \(t \geq 2\) that is maximal with respect to the property

$$\sum_{i=1}^{t} \gamma_i > (t - 1)p.$$

Let $\Gamma_p$ be the set of critical multisets of $\mathbb{F}_p$.

**Example**

- $\Gamma_2 = \emptyset$,
- $\Gamma_3 = \{\{2, 2\}\}$,
- $\Gamma_5 = \{\{2, 4\}, \{3, 3\}, \{3, 4, 4\}, \{4, 4, 4, 4\}\}$, and
- $\Gamma_7 = \{\{2, 6\}, \{3, 5\}, \{4, 4\}, \{3, 6, 6\}, \{4, 5, 6\}, \{5, 5, 5\}, \{4, 6, 6, 6\}, \{5, 5, 6, 6\}, \{5, 6, 6, 6, 6\}, \{6, 6, 6, 6, 6, 6\}\}$. 

Gretchen L. Matthews
Critical multisets play a role in describing a cone containing the pseudocodewords of a nonbinary parity-check code.

Given a prime $p$ and $H \in \mathbb{F}_p^{r \times n}$, consider the cone

$$K_p(H) = \left\{ v \in \mathbb{R}_{\geq 0}^{(p-1)n} : \begin{align*}
\Theta_j(v) &\geq m_i(1) + m_i(2) + \cdots + m_i(p-1), \text{ and} \\
\Theta_j(v) &\geq \sum_{l=1}^{t} m_{\gamma_l \circ a^{-1} \circ h_{ji}^{-1}} n + i_l \\
\forall 1 &\leq j \leq r, i, i_1, \ldots, i_t \in \text{supp (Row}_j(H)), a \in \mathbb{F}_p^*, \\
\{\gamma_1, \ldots, \gamma_t\} &\in \Gamma_p \\
\end{align*} \right\}$$

where $\Theta_j(v) = \frac{1}{p} \sum_{b=1}^{p-1} \left( b \text{Row}_j(H) \right) [m_1(b) \ m_2(b) \ \cdots \ m_n(b)]^T$. 
The fundamental cone is the smallest cone containing the pseudocodewords.

**Lemma (K-M)**

For a prime $p$ and a $p$-ary matrix $H$,

$$\mathcal{K}_p(H) \subseteq K_p(H).$$

If $p = 2, 3$, then equality holds.

**Theorem (K-M)**

Given a $p$-ary parity-check matrix $H$, where $p$ is prime, then

$$\mathcal{P}(H) \subseteq \left\{ v \in K_p(H) : H \mathcal{M}(v)^T \begin{bmatrix} 1 \\ \vdots \\ p - 1 \end{bmatrix} \mod p = 0 \right\}.$$ 

If $p = 2, 3$, then equality holds.
The integer points of the lifted fundamental cone are precisely the pseudocodewords.

**Definition**

Given $H \in \mathbb{F}_p^{r \times n}$ where $p$ is prime, the lifted fundamental cone of $C(H)$ is

$$\hat{\mathcal{K}}_p(H) = \left\{ (\mathbf{m}, \mathbf{a}) \in \mathbb{R}^{(p-1)n+r} \mid \mathbf{v} \in \mathcal{K}_p(H), H \mathcal{M}(\mathbf{v})^T \begin{bmatrix} 1 \\ \vdots \\ p-1 \end{bmatrix} = p\mathbf{a}^T \right\}.$$
The integer points of the lifted fundamental cone are precisely the pseudocodewords.

**Definition**

Given $H \in \mathbb{F}_p^{r \times n}$ where $p$ is prime, the lifted fundamental cone of $C(H)$ is

$$\hat{K}_p(H) = \left\{ (m, a) \in \mathbb{R}^{(p-1)n+r} \mid v \in K_p(H), H \mathcal{M}(v)^T \begin{bmatrix} 1 \\ \vdots \\ p-1 \end{bmatrix} = pa^T \right\}.$$ 

**Theorem (K-M)**

*Given a binary or ternary matrix $H$, there exists a polynomial time algorithm which computes the generating function of the pseudocodewords of $C(H)$ as a finite sum*

$$\sum_{p \in \mathcal{P}(H)} x^p = \sum_i \alpha_i \frac{x^{v_i}}{(1 - x^{w_{i1}}) \cdots (1 - x^{w_{id}})}$$

*where $\alpha_i \in \{0, 1\}$ and $u_i$, $w_{ij}$ are integer vectors for all $i, j$.***

Gretchen L. Matthews
Generating functions obtained via Barvinok’s algorithm enumerate the pseudocodewords.

Example

Consider the binary code $C(H)$ given by a parity check matrix

$$H = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \end{pmatrix}.$$ Then Barvinok 0.27 computes

$$\sum_{p \in \mathcal{P}(H)} x^p = \frac{1-x_1^2 x_2^2 x_3^2 x_4^2}{(1-x_1 x_3 x_4^2)(1-x_1 x_2^2 x_3)(1-x_2 x_3 x_4)(1-x_1 x_2 x_4)(1-x_1 x_3)}$$
Generating functions obtained via Barvinok’s algorithm enumerate the pseudocodewords.

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Consider the binary code $C(H)$ given by a parity check matrix $H = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \end{pmatrix}$. Then Barvinok 0.27 computes

$$\sum_{\mathbf{p} \in \mathcal{P}(H)} x^\mathbf{p} = \frac{1-x_1^2 x_2^2 x_3^2 x_4^2}{(1-x_1 x_3 x_4^2)(1-x_1 x_2^2 x_3)(1-x_2 x_3 x_4)(1-x_1 x_2 x_4)(1-x_1 x_3)}$$

$$= 1 + x_1 x_3 + x_2 x_3 x_4 + x_1 x_2 x_4 + x_1 x_3 x_4^2 + x_1 x_2^2 x_3 + x_1^2 x_3^2 + x_1^2 x_2 x_3 x_4 + x_1 x_2 x_3^2 x_4 + \cdots.$$
Generating functions obtained via Barvinok’s algorithm enumerate the pseudocodewords.

**Example**

Consider the binary code \( C(H) \) given by a parity check matrix

\[
H = \begin{pmatrix}
1 & 1 & 1 & 0 \\
1 & 0 & 1 & 1
\end{pmatrix}
\]

Then Barvinok 0.27 computes

\[
\sum_{p \in \mathcal{P}(H)} x^p = \frac{1-x_1^2 x_2^2 x_3^2 x_4^2}{(1-x_1 x_3 x_4)(1-x_1 x_2 x_3)(1-x_2 x_3 x_4)(1-x_1 x_2 x_4)(1-x_1 x_3)}
\]

\[
= 1 + x_1 x_3 + x_2 x_3 x_4 + x_1 x_2 x_4 + x_1 x_3 x_4^2 + x_1 x_2^2 x_3 + x_1^2 x_3^2 + x_1^2 x_2 x_3 x_4 + x_1 x_2 x_3 x_4^2 + \cdots
\]

Hence, \((0, 0, 0, 0), (1, 0, 1, 0), (0, 1, 1, 1), (1, 1, 0, 1), (1, 0, 1, 2), (1, 2, 1, 0), (2, 0, 2, 0), (2, 1, 1, 1), (1, 1, 2, 1), \ldots\) are among the (unscaled) pseudocodewords of \( C(H) \).
Irreducible pseudocodewords are those pseudocodewords most likely to cause decoder failure.

A nonzero pseudocodeword is irreducible provided it cannot be written as a sum of two or more nonzero pseudocodewords.

Theorem (K-M, K-M)

Let $H \in \mathbb{F}_p^{r \times n}$. Then the set of irreducible pseudocodewords of $C(H)$ is a projection of the Hilbert basis of the lifted fundamental cone of $C(H)$. 
The choice of Tanner graph can impact the number of irreducible pseudocodewords.

**Example**

Recall that $C(H_1) = C(H_2)$ for

\[
H_1 = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 1 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 & 1 & 1 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 1 \\
\end{pmatrix} \quad \quad H_2 = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 1 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 & 1 & 1 & 0 \\
0 & 0 & 1 & 0 & 1 & 1 & 1 \\
\end{pmatrix}
\]
The choice of Tanner graph can impact the number of irreducible pseudocodewords.

Example

Recall that $C(H_1) = C(H_2)$ for

$$H_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 \end{pmatrix} \quad H_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{pmatrix}$$

However, the sets of noncodeword irreducible pseudocodewords for $H_1$ and $H_2$ differ, as shown below.
The choice of Tanner graph can impact the number of irreducible pseudocodewords.

**Example**

If \( H_1 = \begin{bmatrix} 1 & 2 & 2 & 1 \\ 2 & 0 & 1 & 2 \end{bmatrix} \) and \( H_2 = \begin{bmatrix} 0 & 2 & 0 & 0 \\ 2 & 0 & 1 & 2 \end{bmatrix} \), then \( C(H_1) = C(H_2) \).

However, the irreducible pseudocodewords of \( C(H_1) \) and \( C(H_2) \) are

\[
\text{Irr} \ (H_1) = \left\{ \begin{array}{c}
(0, 0, 0, 0, 0, 0, 1),
(0, 0, 1, 1, 0, 0, 0),
(0, 0, 0, 1, 1, 3, 1),
(0, 1, 1, 0, 1, 0, 1),
(0, 3, 1, 0, 1, 0, 1, 2),
(0, 3, 1, 1, 2, 0, 0, 1),
(1, 0, 0, 0, 0, 3, 1, 2),
(1, 1, 0, 1, 0, 3, 2, 0),
(1, 1, 0, 1, 1, 3, 2, 0),
(1, 1, 0, 1, 0, 1, 0, 1),
(1, 1, 0, 1, 1, 0, 1, 0),
(1, 3, 0, 1, 0, 1, 0, 1),
(1, 3, 1, 0, 1, 0, 1, 0),
(1, 3, 1, 1, 0, 0, 0, 1),
(2, 0, 0, 1, 0, 3, 1, 1),
(2, 0, 0, 1, 0, 3, 2, 0),
\end{array} \right\} 
\]

and

\[
\text{Irr} \ (H_2) = \left\{ \begin{array}{c}
(0, 0, 0, 0, 0, 0, 1, 1),
(0, 0, 0, 0, 1, 0, 1, 0),
(0, 0, 0, 1, 1, 0, 0, 0),
(0, 0, 0, 1, 0, 0, 0, 1),
(0, 0, 0, 1, 0, 0, 0, 1),
(0, 0, 0, 1, 0, 0, 0, 0),
(0, 0, 0, 1, 0, 0, 0, 0),
(0, 0, 0, 1, 0, 0, 0, 0),
(0, 0, 1, 0, 1, 0, 0, 0),
(0, 0, 1, 0, 0, 1, 0, 0),
(1, 0, 0, 0, 0, 0, 0, 1),
(1, 0, 0, 0, 0, 0, 0, 0),
(1, 0, 0, 0, 0, 0, 0, 1),
(1, 0, 0, 0, 0, 0, 0, 1),
\end{array} \right\} .
\]


Thank you!