Parity-check codes and their representations

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CLEMSON MATHEMATICAL SCIENCES

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Linear codes can be represented by parity-check matrices.

Let $\mathbb{F}_q$ be the finite field with q elements, where q is a power of a prime.
A linear code C of length n over $\mathbb{F}_q$ is a subspace of $\mathbb{F}_q^n$.
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The matrix H is called a parity-check matrix for C, because

$$y \in C \text{ if and only if } Hy^T = 0.$$

The code C is sometimes called a parity-check code. If H is sparse, then C is a low-density parity-check (LDPC) code.

A parity-check matrix for a linear code is not unique.

Example

Consider the binary code

$$C = \left\{ \begin{array}{l} (0,0,0,0,0,0,0),(0,0,1,0,1,1,1),(0,1,0,1,0,1,1),(0,1,1,1,1,0,0), \\ (1,0,0,1,1,0,1),(1,0,1,1,0,1,0),(1,1,0,0,1,1,0),(1,1,1,0,0,0,1) \end{array} \right\} \subseteq \mathbb{F}_2^7.$$

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Then C has

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 \end{pmatrix}$$

as a parity-check matrix.

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as a parity-check matrix. Another parity-check matrix for C is

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{pmatrix}.$$

A code can be represented by a Tanner graph.

The Tanner graph of a binary matrix H is a bipartite graph $T(H) = (X \cup F, E)$ with biadjacency matrix H, meaning

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Example

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Example

check node bit node

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- $\{x_i, f_j\} \in E$ if and only if $h_{ji} = 1$.

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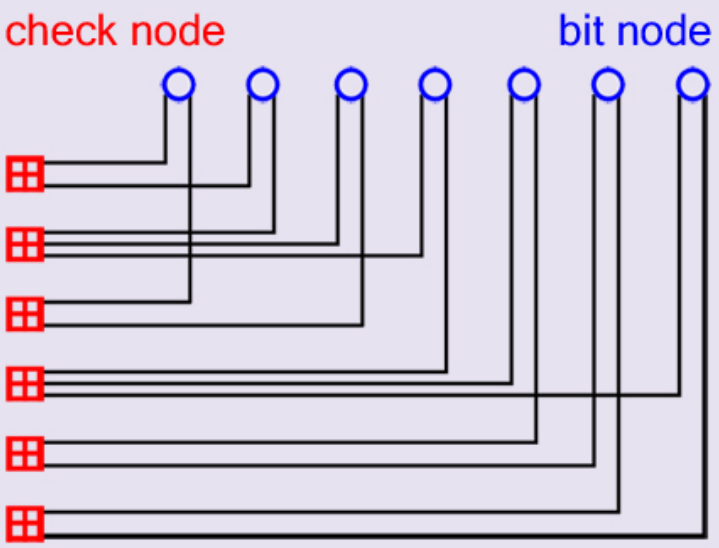
Tanner graph of H :



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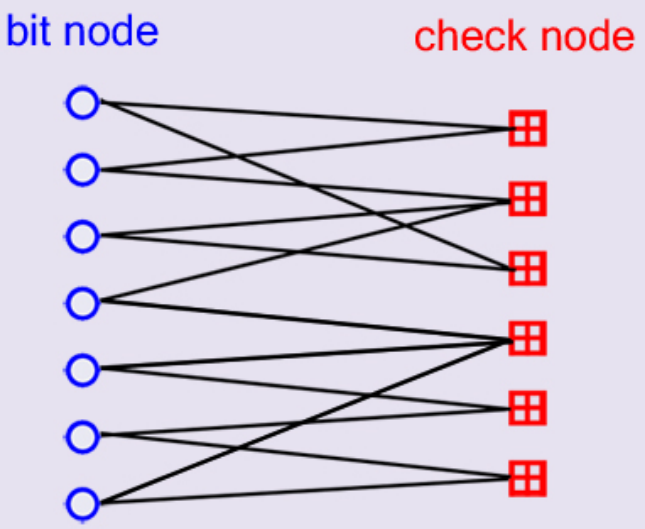


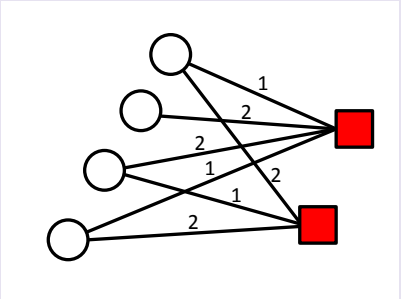
Tanner graphs of q-ary codes are weighted bipartite graphs.

The Tanner graph $T(H)$ of $H \in \mathbb{F}_q^{r \times n}$ is a weighted bipartite graph with biadjacency matrix H; that is,

- the vertex set of $T(H)$ is $\{x_1, \ldots, x_n\} \cup \{f_1, \ldots, f_r\}$,
- the edge set of $T(H)$ is $\{\{x_i, f_j\} : h_{ji} \neq 0\}$, and
- edge weights are given by $wt\{x_i, f_j\} = h_{ji}$.

Example

The Tanner graph of $H = \begin{bmatrix} 1 & 2 & 2 & 1 \\ 2 & 0 & 1 & 2 \end{bmatrix}$ over $\mathbb{F}_3$ is



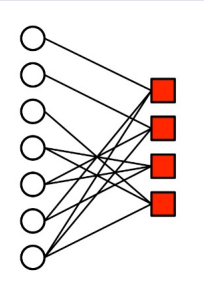
A code may have more than one Tanner graph.

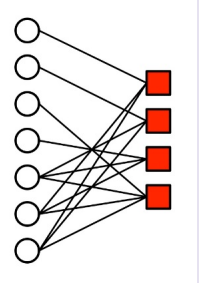
The Tanner graph of a code depends on the choice of parity-check matrix.

Example

Tanner graphs $T(H_1)$ and $T(H_2)$ for the same binary code are shown below.

$$H_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 \end{pmatrix} \quad H_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{pmatrix}$$





To emphasize that a code C is being considered with a fixed parity-check matrix H, we write

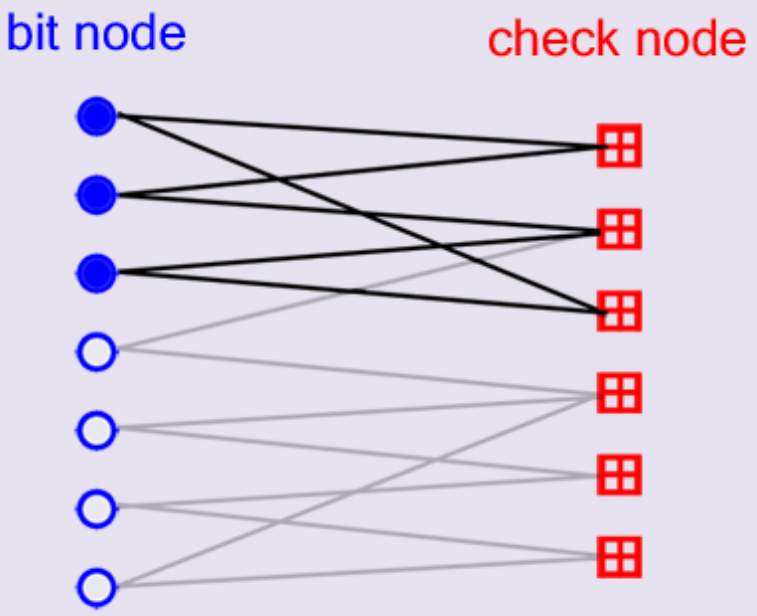
$$C = C(H).$$

Codewords may be represented graphically.

An element $c = (c_1, c_2, \ldots, c_n)$ is a codeword of a binary code $C = C(H)$ if and only if the binary assignment $(c_1, c_2, \ldots, c_n)$ to the bit nodes of $T(H)$ make the binary sum at every check node 0.

Example

$(1, 1, 1, 0, 0, 0, 0)$
corresponds to



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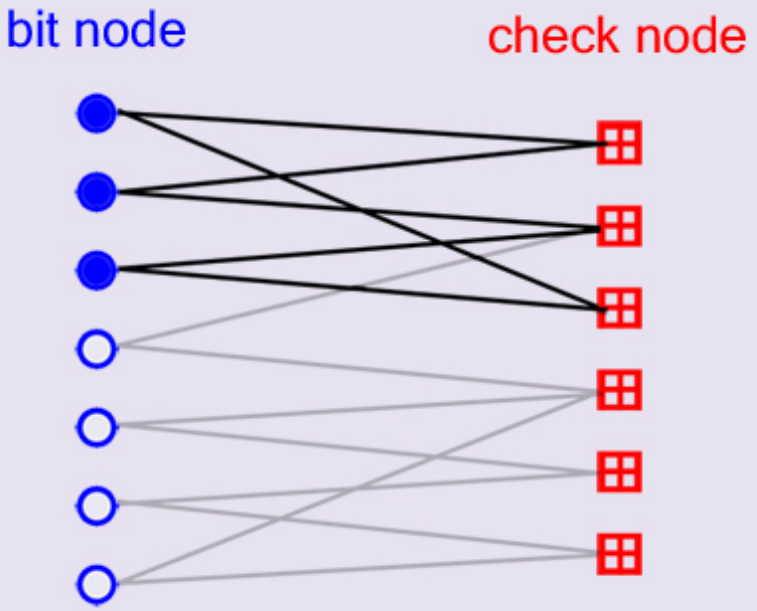

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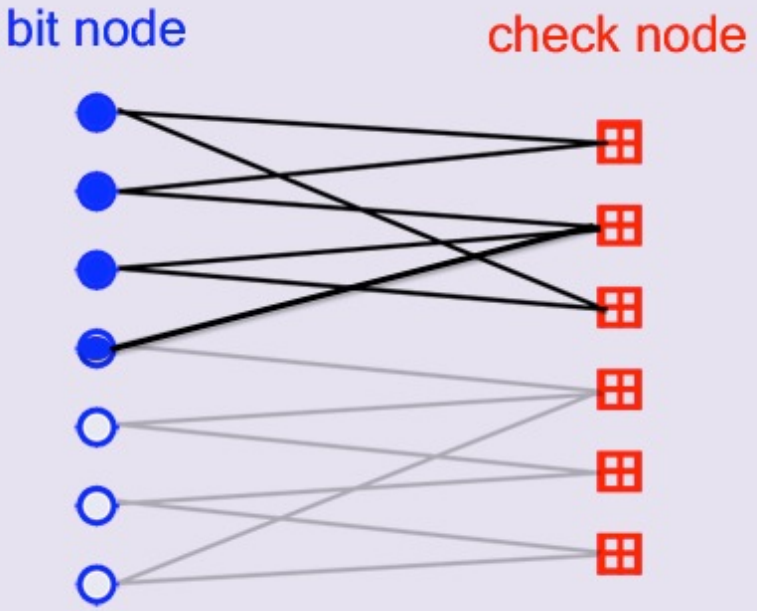

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However, $(1, 1, 1, 1, 0, 0, 0)$ is not a codeword.

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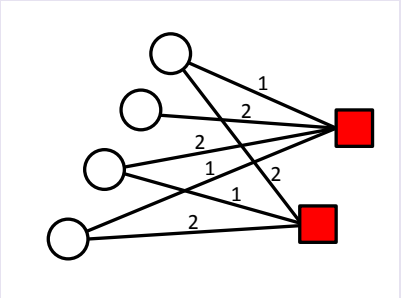

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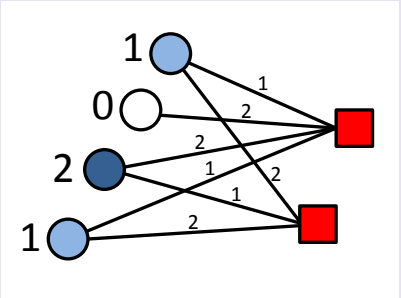

Note that $(1, 0, 2, 1)$ is a codeword of the code defined by H.

Tanner graphs have played various roles in coding theory.

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- the recursive construction of codes and
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The Hamming distance between two words $w, y \in \mathbb{F}_q^n$ is

$$d(w, y) := \#\{i : w_i \neq y_i\}.$$

Code parameters

Classically, the quality of a code C is measured by parameters

- n, the length of the code C
- $k := \dim_{\mathbb{F}_q} C$, the dimension of the code C
- $d := \min\{d(c, c') : c, c' \in C, c \neq c'\}$, the minimum distance of C

A code C with these parameters is called an $[n, k, d]$ code.

Given a received word, a decoder seeks to find the most likely codeword sent.

Minimum distance decoding problem

Given a received word $w \in \mathbb{F}_q^n$, find $y \in C$ such that

$$d(w, y) \leq d(w, c) \;\; \forall c \in C.$$

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Maximum likelihood (ML) decoding problem

Given a received word $w \in \mathbb{F}_q^n$, find $y \in C$ that maximizes

$$P\left(y \textit{ sent} | w \textit{ received}\right),$$

the probability that the codeword y was sent given that w is received.

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Maximum likelihood decoding is equivalent to minimum distance decoding (provided the channel has error probability $< .5$).

The ML decoding problem can be stated as an LP.

Definition

Given a binary code C of length n,

$$poly(C) := \left\{ \sum_{\mathbf{y} \in C} \lambda_{\mathbf{y}} \mathbf{y} : \lambda_{\mathbf{y}} \geq 0, \sum_{y \in C} \lambda_y = 1 \right\} \subseteq [0,1]^n$$

denotes the codeword polytope of C.

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Let

$$\gamma_i := log \left(\frac{P\left(w_i | y_i = 0\right)}{P\left(w_i | y_i = 1\right)} \right)$$

denote the negative log-likelihood ratio at the ith coordinate.

The ML decoding problem is equivalent to the following linear program (LP):

$$\text{minimize} \sum_{i=1}^{n} \gamma_i f_i \text{ subject to } f \in poly(C).$$

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Write $\mathbb{F}_q = \{0, \alpha, \ldots, \alpha^{q-1}\}$, and define $\begin{array}{rccc} \phi: & \mathbb{F}_q^n & \to & \{0,1\}^{(q-1)n} \\ & x & \mapsto & (\mathbf{e}_{x_1}, \mathbf{e}_{x_2}, \ldots, \mathbf{e}_{x_n}). \end{array}$

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Given a code C of length n over $\mathbb{F}_q$, the codeword polytope of C is

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The ML decoding problem for C is equivalent to the following LP:

$$\text{minimize} \sum_{k=1}^{(q-1)n} \gamma_k f_k \text{ subject to } f \in poly(C),$$

where $\gamma_{(q-1)(i-1)+j} := \log\left(\frac{P(y_i|0)}{P(y_i|\alpha^j)}\right)$ is a log-likelihood ratio at the ith coordinate and $j = 1, \ldots, q-1$.

LP relaxation of ML decoding is more managable.

Definition

Suppose $C = C(H)$ be a code over $\mathbb{F}_q$ of length n and $Row_j(H)$ denote the jth row of H. The fundamental polytope of H is

$$Q(H) = \cap_{j=1}^{r} poly\left(C\left(Row_j(H)\right)\right) \subseteq [0,1]^{(q-1)n}.$$

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Example

Consider again

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Linear Code Linear Program [Feldman, Wainwright, & Karger, 2005; Flanagan, Skachek, Byrne, & Greferath, 2009]

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Definition

The vertices of $Q(H)$ are called pseudocodewords.

LP relaxation may lead to a noncodeword output, called a pseudocodeword.

Some, but not necessarily all, pseudocodewords are codewords.

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Theorem [Etzion, Trachtenberg, & Vardy, 1999]

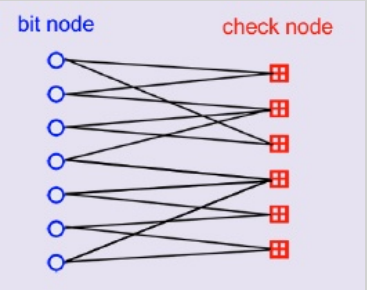
Given an $[n, k, d]$ code with acyclic Tanner graph, either $\frac{k}{n} \leq 0.5$ or

$$\frac{k}{n} > 0.5 \text{ and } d = 2.$$

Pseudocodewords can also be defined via graph covers.

Example

Consider the Tanner graph $T(H)$:



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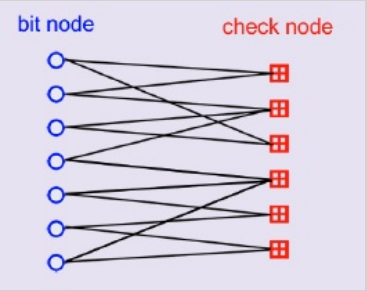
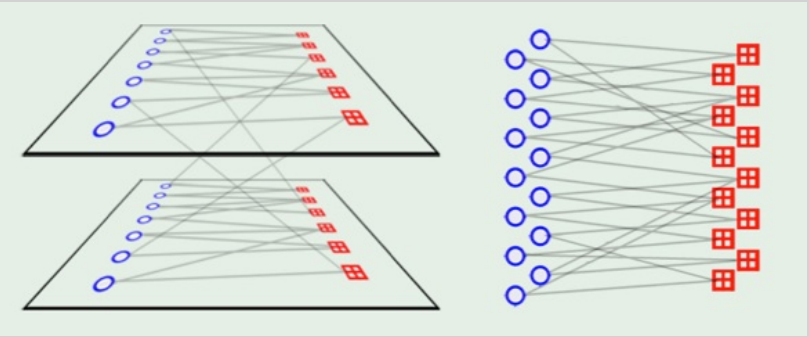


A degree 2 cover of $T(H)$ is



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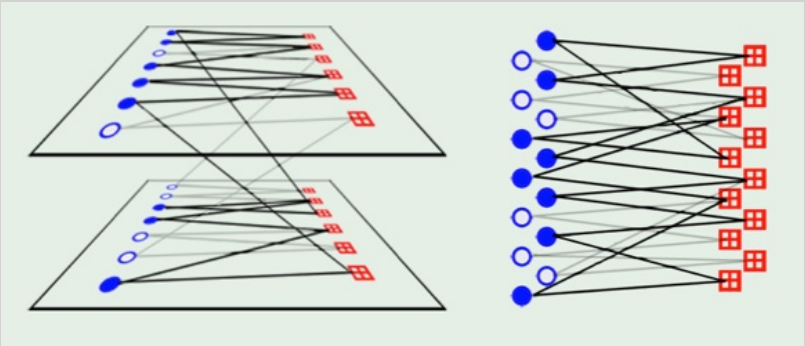
The degree 2 cover gives rise to codewords:

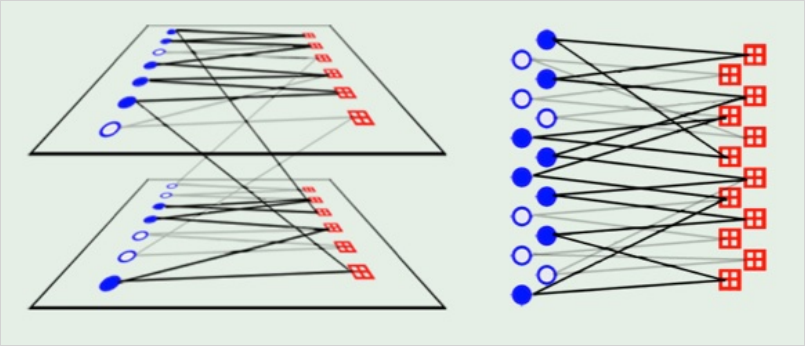


In this case, we obtain $\frac{1}{2}(1,1,1,2,1,1,1)$.

Pseudocodewords can also be defined via graph covers.

Example

The degree 2 cover gives rise to codewords:



Consider

$$\frac{1}{M}(m_1(1), \ldots, m_n(1)) \in [0,1]^n$$

where $m_i(b)$ is the number of copies of the ith symbol that take the value $b \in \mathbb{F}_2$ in a codeword c from $C(\tilde{T}(H))$
In this case, we obtain $\frac{1}{2}(1,1,1,2,1,1,1)$.

Pseudocodewords can also be defined via graph covers.

Let $C = C(H)$ be a code over $\mathbb{F}_q$ of length n.
Pseudocodewords from a degree M graph cover $\tilde{T}(H)$ of $T(H)$ are of the form

$$\frac{1}{M}(m_1(\alpha), \ldots, m_n(\alpha), \ldots, m_1(\alpha^{q-1}), \ldots, m_n(\alpha^{q-1})) \in [0,1]^{(q-1)n}$$

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Pseudocodewords can also be defined via graph covers.

Example

The ternary code $C(H)$ with $H = \begin{bmatrix} 1 & 2 & 2 & 1 \\ 2 & 0 & 1 & 2 \end{bmatrix} \in \mathbb{F}_3^{2\times 4}$

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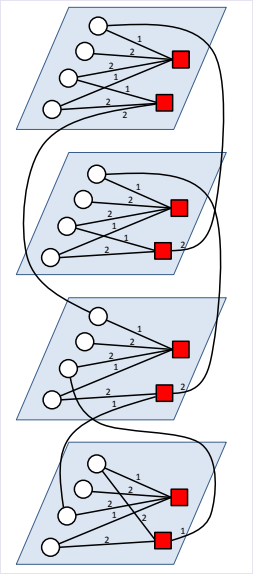
The ternary code $C(H)$ with $H = \begin{bmatrix} 1 & 2 & 2 & 1 \\ 2 & 0 & 1 & 2 \end{bmatrix} \in \mathbb{F}_3^{2\times 4}$ has a degree 4 graph cover of $T(H)$



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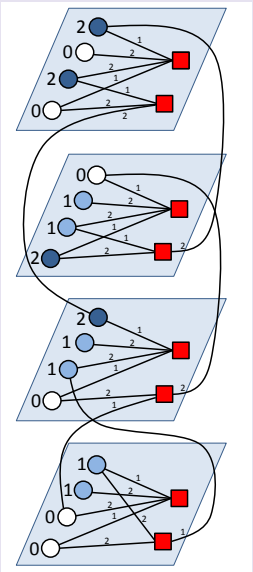
$$\tilde{c} = (2, 0, 2, 1, 0, 1, 1, 1, 2, 1, 1, 0, 0, 2, 0, 0)$$



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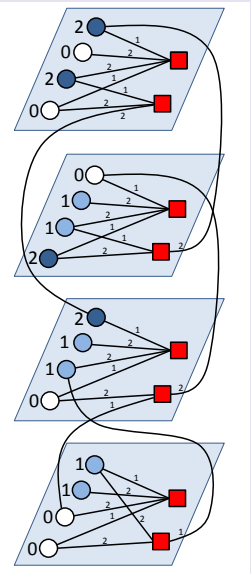

that gives rise to the pseudocodeword $\left(\frac{1}{4}, \frac{3}{4}, \frac{2}{2}, 0, \frac{2}{2}, 0, \frac{1}{4}, \frac{1}{4}\right)$.

LP decoding gives an approximation to iterative decoding algorithms.

Iterative decoders make use of local computations, while ML decoders search for a configuration that satisfies all check nodes.

Advantages of an iterative decoder:

- Can correct more errors than guaranteed by ML decoding
- Does not require any particular structure
- Fast (linear in code length for sparse matrices) and easy to implement

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Aim

To provide tools for determining the expected performance of a parity-check code and for selecting parity-check matrices which lessen the impact of noncodeword pseudocodewords.

Two tools for studying pseudocodewords: generating functions and the fundamental cone.

Notation

Given a matrix H, let $\mathcal{P}(H)$ denote the set of pseudocodewords of $C(H)$

Generating function of the pseudocodewords

Given a code $C = C(H)$, we consider

$$\sum_{p \in \mathcal{P}(H)} x^p.$$

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Fundamental cone

The fundamental cone $\mathcal{K}(H)$ of a code $C(H)$ of length n over $\mathbb{F}_q$ is the smallest cone in $\mathbb{R}^{(q-1)n}$ that contains all pseudocodewords of $C(H)$.

Two tools for studying pseudocodewords: generating functions and the fundamental cone.

Theorem (Koetter, Li, Vontobel, & Walker 2007)

Let $H \in \mathbb{F}_2^{r \times n}$. Given $\mathbf{p} \in \mathbb{Z}^n$, the following are equivalent:

1. $\mathbf{p}$ is a psuedocodeword of $C(H)$ and
2. $\mathbf{p} \in \mathcal{K}(H)$ and $H\mathbf{p}^T = \mathbf{0} \in \mathbb{F}_2^r$.

Theorem (Koetter, Li, Vontobel, & Walker 2007)

Suppose $H \in \mathbb{F}_2^{r \times n}$ is a matrix with exactly two 1's in each column. Then the generating function of the pseudocodewords of $C(H)$ is

$$\sum_{\mathbf{p} \in \mathcal{P}(H)} \mathbf{x}^{\mathbf{p}} = \zeta_{N(H)}(\mathbf{x}),$$

the edge zeta function of the normal graph of $T(H)$.

The generating function for the pseudocodewords of a code is a rational function.

Theorem (K-M)

Given a binary matrix $H \in \mathbb{F}_2^{r \times n}$,

- *the generating function for the pseudocodewords of $C(H)$, $\sum_{p \in \mathcal{P}(H)} x^p$, is a rational function; and*

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- *there exists a polynomial time algorithm which computes the generating function of the pseudocodewords, $\sum_{\mathbf{p} \in \mathcal{P}(H)} \mathbf{x}^{\mathbf{p}}$, as a finite sum*

$$\sum_i \alpha_i \frac{\mathbf{x}^{\mathbf{v}_i}}{(1 - \mathbf{x}^{\mathbf{w}_{i1}}) \dots (1 - \mathbf{x}^{\mathbf{w}_{id}})}$$

where $\alpha_i \in \{0, 1\}$ and $\mathbf{u}_i$, $\mathbf{w}_{ij}$ are integer vectors for all i, j.

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Recall that the fundamental cone of a binary code $C(H)$ is

$$\mathcal{K}(H) := \left\{ v \in \mathbb{R}^n : Row_j H v^T - 2h_{ji} v_i \geq 0, v_i \geq 0 \ \forall 1 \leq i \leq n, 1 \leq j \leq r \right\}.$$

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However, the pseudocodewords of $C(H)$ are those integer points of $\mathcal{K}(H)$ which satisfy $H\mathbf{p}^T = \mathbf{0} \in \mathbb{F}_2^r$.

Lifting the fundamental cone allows us to apply Barvinok's algorithm.

Given $H \in F_2^{r \times n}$, the fundamental cone of $C(H)$ is

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Definition

Given $H \in \mathbb{F}_2^{r \times n}$, the lifted fundamental cone of $C(H)$ is

$$\hat{\mathcal{K}}(H) = \left\{ (\mathbf{v}, \mathbf{a}) \in \mathbb{R}^{n+r} \mid \begin{array}{l} v_i \geq 0, H\mathbf{v}^T = 2\mathbf{a}^T, \text{ and} \\ Row_j H v^T - 2h_{ji} v_i \geq 0 \\ \text{for all } 1 \leq i \leq n \text{ and } 1 \leq j \leq r \end{array} \right\}.$$

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Consider the projection

$$\begin{array}{rccc} \pi : & \mathbb{R}^{n+r} & \rightarrow & \mathbb{R}^n \\ & (\mathbf{v}, \mathbf{a}) & \mapsto & \mathbf{v}. \end{array}$$

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Proposition

Let $H \in \mathbb{F}_2^{r \times n}$. Then $\pi\left(\hat{\mathcal{K}}(H)\right) = \mathcal{K}(H)$ and $\mathcal{P}(H) = \pi\left(\hat{\mathcal{K}}(H) \cap \mathbb{Z}^{n+r}\right)$.

The generating function for the pseudocodewords of a code is a rational function.

Denote the generating function for integer points in the lifted fundamental cone by

$$f(x_1, x_2, \ldots, x_{n+r}) := \sum_{(\mathbf{v},\mathbf{a}) \in \hat{\mathcal{K}}(H) \cap \mathbb{Z}^{n+r}} \mathbf{x}^{(\mathbf{v},\mathbf{a})}.$$

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Then $f(x_1, x_2, \ldots, x_{n+r})$ can be expressed as a rational function. Applying the Barvinok-Woods approach to specialization gives that

$$\sum_{\mathbf{v} \in \mathcal{P}(H)} \mathbf{x}^{\mathbf{v}} = \sum_{\mathbf{v} \in \pi(\hat{\mathcal{K}}(H) \cap \mathbb{Z}^{n+r})} \mathbf{x}^{\mathbf{v}} = \sum_{(\mathbf{v},\mathbf{a}) \in \hat{\mathcal{K}}(H) \cap \mathbb{Z}^{n+r}} \mathbf{x}^{\mathbf{v}} = f(x_1, x_2, \ldots, x_n, 1, 1, \ldots, 1)$$

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Theorem (K-M)

Given $H \in \mathbb{F}_2^{r \times n}$, the generating function for the pseudocodewords of $C(H)$ $\sum_{p \in \mathcal{P}(H)} x^p$ is a rational function.

Critical multisets play a key role in understanding pseudocodewords.

Definition

Given a prime p, a critical multiset of $\mathbb{F}_p$ is a multiset $\{\gamma_1, \ldots, \gamma_t\} \subseteq \{0, 1, \ldots, p-1\}$ with $t \geq 2$ that is maximal with respect to the property

$$\sum_{i=1}^{t} \gamma_i > (t-1)p.$$

Let Γ_p be the set of critical multisets of $\mathbb{F}_p$.

Example

- $\Gamma_2 = \emptyset$,
- $\Gamma_3 = \{\{2,2\}\}$,
- $\Gamma_5 = \{\{2,4\},\{3,3\},\{3,4,4\},\{4,4,4,4\}\}$, and
- $\Gamma_7 = \{\{2,6\},\{3,5\},\{4,4\},\{3,6,6\},\{4,5,6\},\{5,5,5\},\{4,6,6,6\},$ $\{5,5,6,6\},\{5,6,6,6,6\},\{6,6,6,6,6,6\}\}$.

Critical multisets play a role in describing a cone containing the pseudocodewords of a nonbinary parity-check code.

Given a prime p and $H \in \mathbb{F}_p^{r \times n}$, consider the cone

$$K_p(H) = \left\{ v \in \mathbb{R}_{\geq 0}^{(p-1)n} : \begin{array}{l} \Theta_j(v) \geq m_i(1) + m_i(2) + \cdots + m_i(p-1), \text{ and} \\ \Theta_j(v) \geq \sum_{l=1}^{t} m_{\left(\gamma_l \odot a^{-1} \odot h_{ji_l}^{-1} - 1\right) n + i_l} \\ \forall 1 \leq j \leq r, i, i_1, \ldots, i_t \in \text{supp}\left(\text{Row}_j(H)\right), a \in \mathbb{F}_p^*, \\ \{\gamma_1, \ldots, \gamma_t\} \in \Gamma_p \end{array} \right\}$$

where $\Theta_j(v) = \frac{1}{p} \sum_{b=1}^{p-1} \left(b \, \text{Row}_j(H) \right) \left[m_1(b) \; m_2(b) \; \cdots \; m_n(b) \right]^T$.

The fundamental cone is the smallest cone containing the pseudocodewords.

Lemma (K-M)

For a prime p and a p-ary matrix H,

$$\mathcal{K}_p(H) \subseteq K_p(H).$$

If $p = 2, 3$, then equality holds.

Theorem (K-M)

Given a p-ary parity-check matrix H, where p is prime, then

$$\mathcal{P}(H) \subseteq \left\{ v \in K_p(H) : H\mathcal{M}(v)^T \begin{bmatrix} 1 \\ \vdots \\ p-1 \end{bmatrix} \mod p = \mathbf{0} \right\}.$$

If $p = 2, 3$, then equality holds.

The integer points of the lifted fundamental cone are precisely the pseudocodewords.

Definition

Given $H \in \mathbb{F}_p^{r \times n}$ where p is prime, the lifted fundamental cone of $C(H)$ is

$$\hat{\mathcal{K}}_p(H) = \left\{ (\mathbf{m}, \mathbf{a}) \in \mathbb{R}^{(p-1)n+r} \,\middle|\, v \in \mathcal{K}_p(H), H\mathcal{M}(v)^T \begin{bmatrix} 1 \\ \vdots \\ p-1 \end{bmatrix} = p\mathbf{a}^T \right\}.$$

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Theorem (K-M)

Given a binary or ternary matrix H, there exists a polynomial time algorithm which computes the generating function of the pseudocodewords of $C(H)$ as a finite sum

$$\sum_{\mathbf{p} \in \mathcal{P}(H)} \mathbf{x}^{\mathbf{p}} = \sum_i \alpha_i \frac{\mathbf{x}^{\mathbf{v}_i}}{(1 - \mathbf{x}^{\mathbf{w}_{i1}}) \ldots (1 - \mathbf{x}^{\mathbf{w}_{id}})}$$

where $\alpha_i \in \{0, 1\}$ and $\mathbf{u}_i$, $\mathbf{w}_{ij}$ are integer vectors for all i, j.

Generating functions obtained via Barvinok's algorithm enumerate the pseudocodewords.

Example

Consider the binary code $C(H)$ given by a parity check matrix $H = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \end{pmatrix}$. Then Barvinok 0.27 computes

$$\sum_{\mathbf{p} \in \mathcal{P}(H)} \mathbf{x}^{\mathbf{p}} = \frac{1 - x_1^2 x_2^2 x_3^2 x_4^2}{(1 - x_1 x_3 x_4^2)(1 - x_1 x_2^2 x_3)(1 - x_2 x_3 x_4)(1 - x_1 x_2 x_4)(1 - x_1 x_3)}$$

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Hence, $(0,0,0,0), (1,0,1,0), (0,1,1,1), (1,1,0,1), (1,0,1,2), (1,2,1,0), (2,0,2,0), (2,1,1,1), (1,1,2,1), \ldots$ are among the (unscaled) pseudocodewords of $C(H)$.

Irreducible pseudocodewords are those pseudocodewords most likely to cause decoder failure.

A nonzero pseudocodeword is irreducible provided it cannot be written as a sum of two or more nonzero pseudocodewords.

Theorem (K-M, K-M)

Let $H \in \mathbb{F}_p^{r \times n}$. Then the set of irreducible pseudocodewords of $C(H)$ is a projection of the Hilbert basis of the lifted fundamental cone of $C(H)$.

The choice of Tanner graph can impact the number of irreducible pseudocodewords.

Example

Recall that $C(H_1) = C(H_2)$ for

$$H_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 \end{pmatrix} \quad H_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{pmatrix}$$

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However, the sets of
noncodeword irreducible pseudocodewords for H_1 and H_2 differ, as shown below.

$$\underbrace{\underbrace{\begin{matrix} (0,0,0,2,2,2,2), & (0,3,0,1,2,1,1), \\ (0,0,3,2,1,1,1), & (0,0,1,2,1,1,1), \\ (3,0,0,1,1,2,1), & (0,1,0,1,2,1,1), \\ (2,1,0,1,0,1,1), & (1,2,0,1,1,0,1), \\ (2,0,1,0,1,1,1), & (0,2,1,0,1,1,1), \\ (1,0,0,1,1,2,1), & (0,1,2,1,0,1,1), \\ (1,0,2,1,1,0,1), & \end{matrix}}_{\text{Irreducible pseudocodeword of } C(H_1)} \quad \begin{matrix} (2,0,3,0,1,1,1), & (4,0,0,0,2,2,2), & (0,0,0,4,2,2,2), \\ (0,4,0,0,2,2,2), & (2,0,0,0,2,2,2), & (0,2,0,0,2,2,2), \\ (3,0,0,3,1,2,1), & (1,1,0,2,1,1,0), & (3,3,0,0,1,1,2), \\ (1,1,0,0,1,1,2), & (0,1,0,3,2,1,1), & (1,3,0,0,1,1,2), \\ (3,1,0,0,1,1,2), & (1,0,0,3,1,2,1), & (0,0,0,0,2,2,2), \\ (0,3,0,3,2,1,1), & (1,1,2,0,1,1,0), & (0,0,3,0,1,1,1), \\ (0,2,3,0,1,1,1). & & \end{matrix}}_{\text{Irreducible pseudocodeword of } C(H_2)}$$

The choice of Tanner graph can impact the number of irreducible pseudocodewords.

Example

If $H_1 = \begin{bmatrix} 1 & 2 & 2 & 1 \\ 2 & 0 & 1 & 2 \end{bmatrix}$ and $H_2 = \begin{bmatrix} 0 & 2 & 0 & 0 \\ 2 & 0 & 1 & 2 \end{bmatrix}$, then $C(H_1) = C(H_2)$.
However, the irreducible pseudocodewords of $C(H_1)$ and $C(H_2)$ are

$$Irr\,(H_1) = \left\{ \begin{array}{llll}
(0,0,0,0,0,0,1,1), & (0,0,0,0,1,0,1,0), & (0,0,0,1,1,0,0,0), & (0,0,1,0,1,0,0,1), \\
(0,0,1,1,0,0,0,0), & (1,0,0,0,0,0,0,1), & (1,0,0,1,0,0,1,0), & (1,0,1,0,0,0,0,0), \\
(0,0,0,1,1,3,1,1), & (0,0,0,1,2,3,1,0), & (0,0,1,0,1,3,0,1), & (0,0,1,1,0,3,1,1), \\
(0,0,1,1,1,3,1,0), & (0,1,0,0,0,1,1,1), & (0,1,0,0,1,1,1,0), & (0,1,0,1,1,1,0,0), \\
(0,1,1,0,1,1,0,1), & (0,1,1,1,0,1,0,0), & (0,3,0,0,1,0,2,1), & (0,3,0,1,1,0,1,1), \\
(0,3,1,0,1,0,1,2), & (0,3,1,0,2,0,1,1), & (0,3,1,1,0,0,1,1), & (0,3,1,1,1,0,1,0), \\
(0,3,1,1,2,0,0,1), & (0,3,1,2,1,0,0,0), & (0,3,2,0,2,0,0,2), & (0,3,2,1,1,0,0,1), \\
(1,0,0,0,0,3,1,2), & (1,0,0,0,1,3,1,1), & (1,0,0,1,0,3,2,1), & (1,0,0,1,1,3,0,1), \\
(1,0,0,1,1,3,2,0), & (1,0,0,2,1,3,1,0), & (1,0,1,0,0,3,1,1), & (1,0,1,0,1,3,1,0), \\
(1,0,1,1,0,3,0,1), & (1,0,1,1,1,3,0,0), & (1,0,1,2,0,3,1,0), & (1,0,2,1,0,3,0,0), \\
(1,1,0,0,0,1,0,1), & (1,1,0,1,0,1,1,0), & (1,1,1,0,0,1,0,0), & (1,3,0,0,1,0,1,1), \\
(1,3,0,1,0,0,1,0), & (1,3,0,1,1,0,0,1), & (1,3,1,0,0,0,1,1), & (1,3,1,0,1,0,0,2), \\
(1,3,1,0,1,0,1,0), & (1,3,1,1,0,0,0,1), & (1,3,1,1,1,0,0,0), & (1,3,2,0,1,0,0,1), \\
(2,0,0,1,0,3,1,1), & (2,0,0,2,0,3,2,0), & (2,0,1,1,0,3,1,0), & (2,3,1,0,0,0,0,1).
\end{array} \right\}$$

and

$$Irr\,(H_2) = \left\{ \begin{array}{llll}
(0,0,0,0,0,0,1,1), & (0,0,0,0,1,0,1,0), & (0,0,0,1,1,0,0,0), & (0,0,1,0,1,0,0,1), \\
(0,0,1,1,0,0,0,0), & (1,0,0,0,0,0,0,1), & (1,0,0,1,0,0,1,0), & (1,0,1,0,0,0,0,0)
\end{array} \right\}.$$

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Thank you!