Graphs with second largest eigenvalue at most 1

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with X.-M. Cheng and J. H. Koolen.
Which graphs have second largest eigenvalue at most $-1$?
Graphs with second largest eigenvalue at most $-1$

Which graphs have second largest eigenvalue at most $-1$?

- Let $\Gamma$ be an $n$-vertex graph.

- Eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. 
Graphs with second largest eigenvalue at most $-1$

Let $\Gamma$ be a connected graph on $n \geq 2$ vertices with second largest eigenvalue at most $-1$.

- The 2-vertex disconnected graph has spectrum $\{[0]^2\}$.

- Interlacing: $\lambda_i \geq \mu_i$ for $i \in \{1, \ldots, m\}$
  $\implies$ every pair of vertices in $\Gamma$ must be adjacent.

- Hence $\Gamma$ must be complete.
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- The 2-vertex disconnected graph has spectrum $\{0^2\}$.
- Interlacing: $\lambda_i \geq \mu_i$ for $i \in \{1, \ldots, m\} \implies$ every pair of vertices in $\Gamma$ must be adjacent.
- Hence $\Gamma$ must be complete.

Theorem (Smith 1970)

Let $\Gamma$ be a connected graph with second largest eigenvalue at most 0. Then $\Gamma$ is complete multipartite.
Graphs with small second largest eigenvalue

Let $S(b)$ denote the set of connected graphs with second largest eigenvalue at most $b$.

- Cao and Yuan 1993: $S(1/3)$.
- Petrović 1993: $S(\sqrt{2} - 1)$.
- Cvetković and Simić 1995: $S((\sqrt{5} - 1)/2)$. 
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- Cvetković and Simić 1995: $S\left(\frac{\sqrt{5} - 1}{2}\right)$.

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- Li and Yang 2011: Quadcyclic graphs.
Graphs with small second largest eigenvalue

Partial characterisations for $S(1)$.

- Li and Yang 2011: Quad-Tricyclic graphs.
Plan

Classify graphs $\Gamma$ with second largest eigenvalue at most 1 such that $\Gamma$ has precisely three distinct eigenvalues.

- Graphs with three eigenvalues 101.
- Main theorem.
- A structural tool for the proof.
- Idea for the finite search.
- Closing remarks.
Graphs with three eigenvalues

Let $\Gamma$ be a connected graph $(V, E)$ with eigenvalues $\theta_0 > \theta_1 > \theta_2$. Then

$$(A - \theta_1 I)(A - \theta_2 I) = \alpha \alpha^\top.$$
Graphs with three eigenvalues

Let $\Gamma$ be a connected graph $(V, E)$ with eigenvalues $\theta_0 > \theta_1 > \theta_2$. Then

$$A^2 = (\theta_1 + \theta_2)A - \theta_1 \theta_2 I + \alpha \alpha^\top, \quad A\alpha = \theta_0 \alpha.$$ 

$$d_x = -\theta_1 \theta_2 + \alpha_x^2,$$

$$\nu_{x,y} = (\theta_1 + \theta_2)A_{x,y} + \alpha_x \alpha_y.$$ 

- Diameter of $\Gamma$ is 2.

- $\theta_1 \geq 0$ and $\theta_2 \leq -\sqrt{2}$. 
Regular graphs

- Regular graphs with three eigenvalues.  
  **Strongly regular graphs**

- Regular graphs with second largest eigenvalue 1.  
  Complement of graphs with smallest eigenvalue $-2$.

- Regular graphs with three eigenvalues and second largest eigenvalue 1.  
  Complement of strongly regular graphs with smallest eigenvalue $-2$.


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Nonregular graphs

Theorem
Let $\Gamma$ be a connected nonregular graph with three distinct eigenvalues $\theta_0 > \theta_1 > \theta_2$ and $\theta_1 = 1$. Then $\theta_2 = -2$, and $\Gamma$ is the Petersen cone or the Van Dam-Fano graph.

Petersen cone

Van Dam-Fano graph
Main theorem

Theorem

Let $\Gamma$ be a connected graph with three distinct eigenvalues and second largest eigenvalue at most 1. Then $\Gamma$ is one of the following graphs.

(a) A complete bipartite graph;
(b) The Petersen cone;
(c) The Van Dam-Fano graph;
(d) A complete multipartite regular graph;
(e) The complement of a Seidel SRG.
Structure of the proof

Goal: find connected 3-eigenvalue graphs $\Gamma$ with $\theta_1 \leq 1$.

- Reduce to the case where $\Gamma$ has second largest eigenvalue precisely 1. $\implies$ all eigenvalues are integers.

- Reduce to the case where $\Gamma$ has at least three distinct valencies.
  - Regular case follows from Seidel (1968).
  - Biregular case [Cheng, Gavrilyuk, GG, Koolen (2015+)].

- Reduce to the case where $\Gamma$ is not a cone.

- Reduce to the case where the smallest eigenvalue of $\Gamma$ is at least $-29$. 
A structural lemma

Lemma
Let $\Gamma$ be a connected graph with second largest eigenvalue 1. For $x \sim y$, let $\pi$ be a vertex partition with cells $C_1 = \{x, y\}$, $C_2 = \{z \in V(\Gamma) \setminus C_1 \mid z \sim x \text{ or } z \sim y\}$, and $C_3 = \{z \in V(\Gamma) \mid z \not\sim x \text{ and } z \not\sim y\}$. Then the induced subgraph on $C_3$ has maximum degree 1.
A bound for $n$

Lemma

Let $\Gamma$ be a connected $n$-vertex graph with three eigenvalues and second largest eigenvalue 1. Let $m$ denote the multiplicity of the smallest eigenvalue of $\Gamma$. Suppose $x \sim y$. Then $n \leq d_x + d_y - \nu_{x,y} + 2m$. 

Proof. $|C_3| = n - (d_x + d_y - \nu_{x,y})$; $|C_3| \leq 2m$. . . . . .
A bound for $n$

Lemma
Let $\Gamma$ be a connected $n$-vertex graph with three eigenvalues and second largest eigenvalue 1. Let $m$ denote the multiplicity of the smallest eigenvalue of $\Gamma$. Suppose $x \sim y$. Then $n \leq d_x + d_y - \nu_{x,y} + 2m$.

Proof.

$$|C_3| = n - (d_x + d_y - \nu_{x,y}); \quad |C_3| \leq 2m.$$
Finite search

Let $\Gamma$ be a connected $n$-vertex graph with eigenvalues $s > 1 > -t$ and suppose $-t$ has multiplicity $m$. ($\Gamma$ not a cone.)

- $n \leq f(t)$ for some rational function $f$.

- For each $t \in \{3, \ldots, 29\}$, we can enumerate parameters $(n, s, m)$. Denote their set by $S(t)$. 


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| $t$ | $|S(t)|$ | $t$ | $|S(t)|$ | $t$ | $|S(t)|$ |
|-----|--------|-----|--------|-----|--------|
| 3   | 128    | 12  | 497    | 21  | 189    |
| 4   | 196    | 13  | 455    | 22  | 163    |
| 5   | 277    | 14  | 409    | 23  | 143    |
| 6   | 375    | 15  | 377    | 24  | 118    |
| 7   | 492    | 16  | 340    | 25  | 95     |
| 8   | 610    | 17  | 311    | 26  | 76     |
| 9   | 748    | 18  | 273    | 27  | 61     |
| 10  | 898    | 19  | 248    | 28  | 43     |
| 11  | 546    | 20  | 220    | 29  | 27     |
Finite search

- $n \leq f(t)$ for some rational function $f$.

- For each $t \in \{3, \ldots, 29\}$, we can enumerate parameters $(n, s, m)$. Denote their set by $S(t)$.

- For each $S \in S(t)$, we can enumerate valencies $(k_1, \ldots, k_r)$. Denote by $K(t)$.

| $t$ | $|S(t)|$ | $|K(t)|$ | $|S(t)|$ | $|K(t)|$ | $|S(t)|$ | $|K(t)|$ |
|-----|--------|--------|--------|--------|--------|--------|
| 3   | 128    | 58     | 12     | 497    | 287    | 21     | 189    | 137    |
| 4   | 196    | 116    | 13     | 455    | 237    | 22     | 163    | 137    |
| 5   | 277    | 113    | 14     | 409    | 245    | 23     | 143    | 120    |
| 6   | 375    | 173    | 15     | 377    | 214    | 24     | 118    | 104    |
| 7   | 492    | 159    | 16     | 340    | 220    | 25     | 95     | 92     |
| 8   | 610    | 225    | 17     | 311    | 184    | 26     | 76     | 71     |
| 9   | 748    | 233    | 18     | 273    | 190    | 27     | 61     | 59     |
| 10  | 898    | 297    | 19     | 248    | 162    | 28     | 43     | 43     |
| 11  | 546    | 272    | 20     | 220    | 172    | 29     | 27     | 27     |
Finite search

- For each $t \in \{3, \ldots, 29\}$, we can enumerate parameters $(n, s, m)$. Denote their set by $S(t)$.

- For each $S \in S(t)$, we can enumerate valencies $(k_1, \ldots, k_r)$. Denote by $K(t)$.

- For each $S \in S(t)$ and $K \in K(t)$, we can enumerate valency multiplicities $(n_1, \ldots, n_r)$. Denote by $M(t)$.

| $t$ | $|S(t)|$ | $|K(t)|$ | $|M(t)|$ | $t$ | $|S(t)|$ | $|K(t)|$ | $|M(t)|$ | $t$ | $|S(t)|$ | $|K(t)|$ | $|M(t)|$ |
|-----|--------|--------|--------|-----|--------|--------|--------|-----|--------|--------|--------|
| 3   | 128    | 58     | 0      | 12  | 497    | 287    | 0      | 21  | 189    | 137    | 0      |
| 4   | 196    | 116    | 1      | 13  | 455    | 237    | 0      | 22  | 163    | 137    | 0      |
| 5   | 277    | 113    | 2      | 14  | 409    | 245    | 0      | 23  | 143    | 120    | 0      |
| 6   | 375    | 173    | 0      | 15  | 377    | 214    | 0      | 24  | 118    | 104    | 0      |
| 7   | 492    | 159    | 1      | 16  | 340    | 220    | 0      | 25  | 95     | 92     | 0      |
| 8   | 610    | 225    | 0      | 17  | 311    | 184    | 0      | 26  | 76     | 71     | 0      |
| 9   | 748    | 233    | 0      | 18  | 273    | 190    | 0      | 27  | 61     | 59     | 0      |
| 10  | 898    | 297    | 0      | 19  | 248    | 162    | 0      | 28  | 43     | 43     | 0      |
| 11  | 546    | 272    | 0      | 20  | 220    | 172    | 0      | 29  | 27     | 27     | 0      |
Survivors

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</tr>
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</table>

- Use ad-hoc methods to show nonexistence of graphs corresponding to each of the parameters in the table.
Closing remarks

- D. de Caen: must graphs with three eigenvalues have at most three valencies?

- Regular: Strongly regular graphs.

- Bi-regular: Infinitely many examples.

- Tri-regular: Finitely many known examples.

- At least four valencies: No known examples.