A Tiling and (0, 1)-Matrix Existence Problem

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Let $R = (r_1, r_2, \ldots, r_m, r_{m+1} = 0)$ and $S = (s_1, s_2, \ldots, s_n)$ be nonnegative integral vectors.
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Question: Can a \((m + 1) \times n\) checkerboard be tiled with vertical dimers and monomers so that there are \( r_i \) dimers with the upper half of the dimer in row \( i \) and \( s_i \) dimers in column \( i \)?
Example: $R = (2, 2, 1, 2, 0); S = (2, 1, 2, 2)$
A $(0, 1)$-Matrix and Tiling Problem

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2. The existence of a $(0, 1)$-matrix where no consecutive 1’s occur in a column.

3. Phrase it as a linear programming problem and look for a $0, 1$ solution.

\[
\begin{align*}
    a_{11} + a_{12} + \cdots + a_{1n} &= r_1, \\
    \vdots & \quad \vdots \\
    a_{m1} + a_{m2} + \cdots + a_{mn} &= r_m.
\end{align*}
\]
Other Ways to Phrase the Question

1. A question about the existence of a \((0, 1)\)-matrix where every sequence of 1’s in a column has an even number of 1’s.
2. The existence of a \((0, 1)\)-matrix where no consecutive 1’s occur in a column.
3. Phrase it as a linear programming problem and look for a 0, 1 solution. 
\((a_{11} + a_{12} + \cdots + a_{1n} = r_1, \text{ etc.})\)
Our Point of View

The existence of a \((0,1)\)-matrix where no consecutive 1’s occur in a column.
Definition

Let $A(R, S)$ be the set of all $(0, 1)$-matrices with

- row sum vector $R$
- column sum vector $S$. 
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-Studied by H.J. Ryser, D. Gale, D.R. Fulkerson, R.M Haber, and R. Brualdi.
Definition

Let $A_1(R, S)$ be the set of all $(0, 1)$-matrices with
- row sum vector $R$
- column sum vector $S$
- no consecutive 1’s occur in any column.
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- row sum vector $R$
- column sum vector $S$
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Question

When is $A_1(R, S)$ nonempty?
Example

\[ R = (1, 1, 3, 2, 2, 3); \quad S = (3, 1, 3, 1, 1, 3) \]
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Example

\[ R = (1, 1, 3, 2, 2, 3); \quad S = (3, 1, 3, 1, 1, 3) \]
An Observation

Observation: If $M \in A_1(R, S)$ then we can entry wise sum rows $r_i$ and $r_{i+1}$ and get a matrix in $A((r_1, \ldots, r_{i-1}, r_i + r_{i+1}, r_{i+2}, \ldots), S)$. 
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The Gale-Ryser Theorem characterizes when $A(R, S)$ is nonempty.
Majorization:

- Nonincreasing integral vectors: \( a = (a_1, a_2, \ldots, a_n) \) and \( b = (b_1, b_2, \ldots, b_m) \). Append zeros to make them equal length (say \( n \geq m \)).

- \( a \) is majorized by \( b \), denoted \( a \preceq b \) when

\[
a_1 + a_2 + \cdots + a_k \leq b_1 + b_2 + \cdots + b_k \quad \text{for all } k
\]

and

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a_1 + a_2 + \cdots + a_n = b_1 + b_2 + \cdots + b_n
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Example: \((3, 2, 1, 0) \preceq (3, 3, 0, 0)\) since

\[3 \leq 3\]

\[3 + 2 \leq 3 + 3\]

\[3 + 2 + 1 = 3 + 3 + 0\]

\[3 + 2 + 1 + 0 = 3 + 3 + 0 + 0\]
Definition

Conjugate of a nonnegative integral vector:

\[ R = (3, 2, 3, 1) \quad R^* = (4, 3, 2, 0, \ldots, 0) \]
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Existence Theorem for $A(R, S)$

Theorem (Gale-Ryser, 1957)

If $R = (r_1, r_2, \ldots, r_m)$ and $S = (s_1, s_2, \ldots, s_n)$ are nonnegative integral vectors such that $S$ is nonincreasing, then there exists an $m \times n, (0, 1)$-matrix with row sum vector $R$ and column sum vector $S$ if and only if $S \preceq R^*$. 

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For $A_1(R, S)$,

$S \preceq (r_1 + r_2, r_3 + r_4, r_5)^*$

$S \preceq (r_1 + r_2, r_3, r_4 + r_5)^*$

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Let $Q_R = \{(r_1 + r_2, r_3 + r_4, r_5), (r_1 + r_2, r_3, r_4 + r_5), (r_1, r_2 + r_3, r_4 + r_5)\}$. 
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Observation: If $A_1(R, S)$ is nonempty then $S \preceq q^*$ for all $q \in Q_R$. 

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Observation: If $A_1(R, S)$ is nonempty then $S \preceq q^*$ for all $q \in Q_R$.

Is this condition sufficient to show $A_1(R, S)$ is nonempty?
Existence Theorem for $A_1(R, S)$

Theorem (N., Shader)

If $R = (r_1, r_2, \ldots, r_m)$ and $S = (s_1, s_2, \ldots, s_n)$ are nonnegative integral vectors such that $S$ is nonincreasing, then there exists an $m \times n$ $(0,1)$-matrix with no two 1’s occurring consecutively in a column and with row sum vector $R$ and column sum vector $S$ if and only if

$$S \preceq q^* \quad \forall q \in Q_R.$$
-direct combinatorial arguments
-network flows
Proofs of the Gale-Ryser Theorem

- direct combinatorial arguments
- network flows
Network Flow for $A(R, S)$

\[
\begin{bmatrix}
1 & 1 & 0 \\
0 & 0 & 1 \\
1 & 0 & 0
\end{bmatrix}
\]
For $A(R, S)$
Existence Theorem for $A_1(R, S)$

**Proof.**

Main idea: Induction
Existence Theorem for $A_1(R, S)$

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- Induct on $n$ (the number of columns).
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Existence Theorem for $A_1(R, S)$

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  - Let $M$ be a $(0, 1)$-matrix in
    $A_1((r_1, r_2, \ldots, r_i - 1, \ldots, r_m), (s_1, \ldots, s_n - 1))$.  

If there is a 1 in the $(i, n)$ position, argue that with a switch this can be changed to a 0.  
Put a 1 in the $(i, n)$ position and use switches to remove any consecutive 1's.  
This completes the induction on the number of 1's in the last column and in turn the induction on the number of columns.
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Definition

The graph of $A_1(R, S)$ is an undirected graph with:

- vertices are the matrices in $A_1(R, S)$
- $M_1 \sim M_2$ if and only if the matrix $M_1$ can be changed to $M_2$ with one basic switch.
Further Work

- Let $M \in A_1(R, S)$. What is the probability that a 1 occurs in position $m_{ij}$?
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Determine statistical information about $A_1(R, S)$ by studying a Markov chain defined on the graph of $A_1(R, S)$. 

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- What if every 1 is followed by $j$ zeros: $A_j(R, S)$?
- Further study of the network flow connection.
Further Work

$A(R, S)$:

$A_1(R, S)$:

(0, 1)-matrix

Flow

Conditions
[1] R. Brualdi
    *Combinatorial Matrix Classes.*

    *Combinatorial Matrix Theory.*

[3] L. Ford and D. Fulkerson
    Maximal flow through a network.
    *Canadian Journal of Mathematics, 8:399, 1956.*

    *A Course in Combinatorics.*