Examples for symbolic solving
Math 676 (Computational Algebraic Geometry)
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The point of today’s lab is to mess around more with Maple, Singular, and Macaulay 2, particularly with respect to solving polynomial systems. As usual, anything is kosher – chase down the things that interest you.

First, let’s remember some useful commands in each package (not exhaustive lists!). Then we’ll get to the systems to try. Divide and conquer might be the way to go....

**Useful commands**

**Maple**

First, load `Groebner`, `PolynomialIdeals`, and `RootFinding` using `with()`.

\[
\text{I1 := PolynomialIdeal([ListOfPolys], characteristic=0); will create an ideal out of the list of polynomials (in characteristic 0, for example).}
\]

\[
\text{I1 := <ListofPolys>; will probably suffice (and save you a moment over the previous command).}
\]

`EquidimensionalDecomposition(Id);` takes in a PolynomialIdeal Id and produces a list of ideals, one for each dimension of the components of the variety.

`EliminationIdeal(Id, vars);` takes in a PolynomialIdeal Id and a list of variables and returns the elimination ideal consisting of polynomials in the variables vars only. Applying this to the pieces of the output of the previous may be handy....

`Isolate(Id, [vars]);` takes in a PolynomialIdeal Id and a list of variables vars and spits out the isolated solutions (and chokes if their are positive-dimensional components).

`HilbertDimension(I);` tells you the dimension of the variety corresponding to an ideal.

**Singular**

First, do `LIB 'solve.lib';`, `LIB 'general.lib';`, and `LIB 'primdec.lib';`.

\[
\text{ring R=0,(x,y),dp; defines a ring with characteristic 0 in variables x,y and endowed with the grevlex monomial order (for example).}
\]

\[
\text{ideal I = list of polynomials; creates an ideal in the current working ring.}
\]

\[
\text{t=timer; solve(I); t-timer; will churn out the solutions of I (if it is zero-dimensional!), and}
\]

\[
\text{the timer business will give you the total elapsed time in hundredths of seconds.}
\]

\[
\text{ideal G = groebner(Id); G; will find a Gröbner of ideal Id and print it out.}
\]

\[
\text{primdecGTZ(I); will give you the primary decomposition of ideal I.}
\]

**Macaulay 2**

\[
\text{R=QQ[x,y,z] sets R to be the polynomial ring in 3 variables with rational coefficients.}
\]

\[
\text{dim I and deg I spit out what you might imagine they would – the dimension and degree of the corresponding variety.}
\]

\[
\text{J=eliminate(x,I) eliminates the variable x from the ideal I. Unfortunately, the package real-}
\]

\[
\text{roots.m2, which contains methods for solving univariate polynomials, seems to be AWOL, so we}
\]

\[
\text{can’t very well compute the isolated solutions of I using those methods....}
\]
\textbf{Examples}

Here are a bunch of examples to try out. Feel free to use whichever software packages tickles your fancy. Next class, be ready to talk about your impressions of these three packages (even if your tests today went poorly) – the feedback that we share with each other will be helpful to any of us who, down the road, needs to pick a software package to use for a problem.

First, I’ll mention a few classes of examples that would be good for testing how the packages scale. Then I’ll give a few purely zero-dimensional systems, followed by some with known positive-dimensional components.

\textbf{PLEASE NOTE} that many of the examples come from \url{www.math.uic.edu/~jan} (click on Demos(tree)) - Jan Verschelde’s website. For kicks, you could go there and grab a few more systems not on these lists.

\textbf{Classes of examples}

1. Adjacent minor problems. Given an $n \times m$ matrix of indeterminants $x_{i,j}$, write down all $k \times k$ adjacent minors (“adjacent” meaning that the rows and columns are all adjacent). These are positive-dimensional ideals, and the case $m = k = 2$ has an especially nice description. All irreducible components have the same degree, and there is a special number of them which depends on (and grows with) $n$. Can you find a description of that number? (These problems arise when considering the set of $n \times m$ matrices of rank no more than $k-1$. The ideal for that set is generated by the set of all $k \times k$ minors and is prime (yielding an irreducible component). The adjacent minor systems have a subset of the generators of these more general problems, so one of the components in the varieties of the adjacent minor ideals will correspond to the set of matrices of rank no more than $k$.)

2. Cyclic n-roots problems. These have a nice description. Here’s the case $n = 5$:

\[
\begin{align*}
    a + b + c + d + e \\
    ab + bc + cd + de + ea \\
    abc + bcd + cde + dea + eab \\
    abcd + bcde + cdea + deab + eabc \\
    abcde - 1
\end{align*}
\]

In other words, the n-roots problem consists of n polynomials in n variables. The first n-1 each have n terms, with the $i^{th}$ one consisting of the degree $i$ terms that list the variables in order (where we can wrap around from e to a, in the case of n=5). The last polynomial is just the product of all n variables minus 1. The interesting thing here is that the solution set is zero-dimensional if n is prime and positive-dimensional otherwise (this is at least a conjecture known to be true through n=15 or so). The structure of the positive-dimensional solution sets in the non-prime case is an open problem. These can be found on Jan Verschelde’s website.

3. Build your own: Recall that high degrees and long prime factorizations (for the coefficients) cause much wailing and gnashing of teeth for Gröbner basis methods. Try making your own family. For kicks, try to solve these in each package and compare results....
Known zero-dimensional

1. The set of polynomials $x_i^{k_i} - 1$ for $i = 1, ..., N$ (you pick $N$) and any old positive choices of $k_i$ will give you isolated solutions. Specifically, you’ll get tuples of roots of unity.

2. The cyclic n-root problems above have only isolated solutions if $n$ is prime (see Verschelde’s website).

3. The economics/original problems seem to be zero-dimensional, according to Jan’s website.

4. Ditto for mechanisms/kin*.

5. The Butcher problem (under PoSSo Suite on Jan’s website) and Caprasse (in the same place).

Known positive-dimensional

1. Just to get your bearings, try the system $xy = x(x - 1) = 0$. This should give you a line and a point. Maple’s EquidimensionalDecomposition handles this well, for example. In Maple, applying HilbertDimension to each piece of the output tells you the dimension.

2. Here’s a fairly small system from a paper of Sommese-Verschelde-Wampler:

$$\begin{align*}
(y - x^2)(x^2 + y^2 + z^2 - 1)(x - 1/2) \\
(z - x^3)(x^2 + y^2 + z^2 - 1)(y - 1/2) \\
(y - x^2)(z - x^3)(x^2 + y^2 + z^2 - 1)(z - 1/2)
\end{align*}$$

The solution set is a sphere, three lines, a twisted cubic curve, and a point.

3. The equations for the twisted cubic are $y - x^2 = z - x^3 = 0$.

4. The family of adjacent minor systems (described above) have positive-dimensional components.

5. The cyclic n-roots problems described above have positive-dimensional components for $n$ not prime.