Notes on Homework 8

1. Before starting, it’s worth recalling that if \( W \) and \( V \) are vector spaces, and if \( (w_1, \ldots, w_n) \) is a basis, and if \( v_1, \ldots, v_n \) are arbitrarily chosen vectors, then there is a unique linear transformation \( f \in L(W, V) \) such that for each \( i \), \( f(w_i) = v_i \). To see how this is defined, if \( w \in W \), then there is a unique choice of numbers \( a_1, \ldots, a_n \in F \) such that \( w = \sum a_iw_i \); if we set \( f(w) = \sum a_if(w_i) = \sum a_iv_i \), this turns out to be a linear transformation. (Nothing can go wrong, in some sense, since there’s only one way to represent a given \( w \) as a sum of the \( w_i \).)

In this example, note that \( z = x + 2y \).

(a) Suppose there were such a linear transformation \( f \); then

\[
f(z) = f(x + 2y) \\
= f(x) + 2f(y) \\
= \begin{pmatrix} 2 \\ 3 \end{pmatrix} + 2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\
= \begin{pmatrix} 2 \\ 5 \end{pmatrix};
\]

since this doesn’t hold, there is no such linear transformation.

(b) The restrictions are compatible, in that \( f(x) + 2f(y) = f(z) \); there is such a linear transformation.

(c) Check:

\[
f(x) + 2f(y) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} + 2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\
= \begin{pmatrix} 1 \\ 3 \end{pmatrix} \\
\neq \begin{pmatrix} 3 \\ 3 \end{pmatrix},
\]

so no such linear transformation is possible.

2. We need to show that for each \( v \in V \) there are numbers \( a_1, \ldots, a_n \in F \) such that \( v = a_1f(w_1) + \cdots + a_nf(w_n) \). So, suppose \( v \in V \). Since \( f \) is surjective, there is some \( w \in W \) such that \( f(w) = v \). Since \( w_1, \ldots, w_n \) spans \( W \), there are numbers \( a_1, \ldots, a_n \in F \) such that \( w = a_1w_1 + \cdots + a_nw_n \). Then

\[
v = f(w) \\
= f(a_1w_1 + \cdots + a_nw_n) \\
= f(a_1w_1) + \cdots + f(a_nw_n) \text{ additivity} \\
= a_1f(w_1) + \cdots + a_nf(w_n) \text{ homogeneity} \\
\in \text{span}(f(w_1), \ldots, f(w_n)).
\]
3. What follows gives a quick indication of why each statement is true. Since in lecture we gave short
shrift to row spaces, those questions won’t be graded. In each of these questions, I’ll identify the
matrix $A$ with a linear transformation $f \in \mathcal{L}(\mathbb{F}^n, \mathbb{F}^m)$. Recall the dictionary from class:

<table>
<thead>
<tr>
<th>$f$</th>
<th>$A$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\text{im}(f)$</td>
<td>$\text{col}(A)$</td>
</tr>
<tr>
<td>$\text{dim im}(f)$</td>
<td>$\text{rank}(A)$</td>
</tr>
<tr>
<td>$\ker(f)$</td>
<td>$\text{nullspace}(A)$</td>
</tr>
<tr>
<td>$\text{dim ker}(f)$</td>
<td>$\text{nullity}(A)$</td>
</tr>
</tbody>
</table>

Recall the basic result

$$\text{rank}(A) + \text{nullity}(A) = n.$$  \hfill (1)

(a) If $\text{rank}(A) = m$ then the system $Ax = b$ has at least one solution for every $b \in \mathbb{F}^m$. True. 
Since $\text{rank}(A) = m = \text{dim im}(f)$, the image of $f$ is an $m$-dimensional subspace of $\mathbb{F}^m$, 
i.e., the whole thing; so each $b$ is in the image of $f$, and each $b$ admits a solution to $Ax = b$.

(b) If $\text{rank}(A) = n$ then the system $Ax = b$ has at least one solution for every $b \in \mathbb{F}^m$. False, in 
the sense of “not necessarily”; consider the case where $n = 1$ and $m = 2$ (or anything 
larger), so that $A$ looks something like, say, $A = (a_{11}a_{12})$. Suppose that some $a_{1j} \neq 0$. 
Then $A$ has rank $n = 1$. But since the target space has dimension $m > 1$, this means 
that there are things in $\mathbb{F}^m$ which aren’t in the image of $f$.

(c) If $\text{rank}(A) = m$ then the system $Ax = b$ has at most one (i.e., no solution or a unique solution) 
for every $b \in \mathbb{F}^m$. False, in the sense of “not necessarily”. The condition that solutions 
are unique is equivalent to the condition that the kernel is trivial, so that the nullity is 
zero; but especially if $n > m$, it is easy to cook up examples, compatible with $(1)$, in 
which the nullity is positive.

(d) If $\text{rank}(A) = n$ then the system $Ax = b$ has at most one (i.e., no solution or a unique solution) 
for every $b \in \mathbb{F}^m$. True; by $(1)$, if $\text{rank}(A) = n$ then $\text{nullity}(A) = 0$, which is equivalent 
to $f$ being injective (one-to-one).

(e) If $Ax = b$, then $x \in \text{row}(A)$. False, in general; if $x$ is any element of $\mathbb{F}^n$, then it’s the 
solution to $Ax = b$ for some $b \in \mathbb{F}^m$.

(f) If $Ax = b$, then $b \in \text{col}(A)$. True; the column space is the same as the image.

(g) If $B \in \text{Mat}(m, n, \mathbb{F})$ is row equivalent to $A$ then $\text{col}(A) = \text{col}(B)$. False; in solving 
equations, manipulating rows should change the right-hand side of the equation, too.

(h) If $B \in \text{Mat}(m, n, \mathbb{F})$ is row equivalent to $A$ then $\text{row}(A) = \text{row}(B)$. True; the operations 
on rows preserve the span of the vectors.

(i) If the columns of $A$ are linearly independent then the rows of $A$ are linearly independent. False; 
consider the case $n \geq 2$, $m = 1$, in which $A$ consists of a single, nonzero column.

(j) If $\text{col}(A) = \mathbb{F}^m$ and $m \leq n$ then $\text{row}(A) = \mathbb{F}^n$. False; if $m < n$, then the row space is 
spanned by at most $m$ vectors, and thus can’t be all of $\mathbb{F}^n$. 

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4. (a) By definition of the map, to compute $[f(v_1)]_E$ we simply multiply;

\[
[f(v_1)]_E = [f]_{E \rightarrow E} [v_1]_E \]

\[
= \begin{pmatrix} -3 & -8 \\ 10 & 27 \end{pmatrix} \begin{pmatrix} 5 \\ 3 \end{pmatrix} \]

\[
= \begin{pmatrix} -39 \\ 131 \end{pmatrix}.
\]

(b) Let $z_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $z_2 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$. Then

\[
[id]_{E \leftrightarrow C} = ([id(z_1)]_E [id(z_2)])
\]

\[
= \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}
\]

so that

\[
[id]_{C \leftrightarrow E} = ([id]_{E \leftrightarrow C})^{-1} = \begin{pmatrix} 3 & -2 \\ -1 & 1 \end{pmatrix}.
\]

Then we can compute:

\[
[f]_{C \leftrightarrow C} = [id]_{C \leftrightarrow E} [f]_{E \leftrightarrow E} [id]_{E \leftrightarrow C}
\]

\[
= \begin{pmatrix} 3 & -2 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} -3 & -8 \\ 10 & 27 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}
\]

\[
= \begin{pmatrix} -107 & -292 \\ 48 & 131 \end{pmatrix}.
\]

(c)

\[
[f(v_2)]_C = [f]_{C \leftrightarrow C} [v_2]_C
\]

\[
= \begin{pmatrix} -107 & -292 \\ 48 & 131 \end{pmatrix} \begin{pmatrix} 3 \\ 4 \end{pmatrix}
\]

\[
= \begin{pmatrix} -1489 \\ 668 \end{pmatrix}.
\]

5. (a) Let $e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. In each case, to write down the matrix of the relevant function, we need to calculate $[f_{21}(e_1)]_E$ and $[f_{21}(e_2)]_E$. Unfortunately, the picture makes it a little hard to figure out which points on the house correspond to $e_1$ and $e_2$. 

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However, let \( v_1 = \begin{pmatrix} 3 \\ 0 \end{pmatrix} \) and \( v_2 = \begin{pmatrix} 0 \\ 4 \end{pmatrix} \). On one hand, these points on the house are distinctive. On the other hand, by homogeneity, we have
\[
 f_{21}(v_1) = f_{21}(3e_1) = 3f_{21}(e_1) \quad f_{21}(v_1) = \frac{1}{3} f_{21}(e_1)
\]
and similarly
\[
 f_{21}(v_2) = \frac{1}{4} f_{21}(v_2).
\]
Since \( f_{21}(v_1) = \begin{pmatrix} 3 \\ 0 \end{pmatrix} \) and \( f_{21}(v_2) = \begin{pmatrix} 8 \\ 4 \end{pmatrix} \), we have
\[
 [f]_{E\leftarrow E}\begin{pmatrix} 1 \\ 2 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 2 \\ 1 \end{pmatrix}.
\]

(b) Same strategy, different numbers. Without any annotation, the results are:
\[
\begin{align*}
\text{iii. } & [f_{31}]_{E\leftarrow E}\begin{pmatrix} 1 \\ 0 \\ 1 \\ 5/4 \end{pmatrix} \\
\text{iv. } & [f_{41}]_{E\leftarrow E}\begin{pmatrix} -1 \\ 0 \\ 0 \\ 2 \end{pmatrix} \\
\text{v. } & [f_{51}]_{E\leftarrow E}\begin{pmatrix} 1 \\ 9/4 \\ 2 \\ 6/4 \end{pmatrix} \\
\text{vi. } & [f_{61}]_{E\leftarrow E}\begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}.
\end{align*}
\]

6. (a) We did this in class; the results follow from calculus. Specifically, if \( p(z), q(z) \in P_3(\mathbb{R})[z] \), then
\[
 D(p(z) + q(z)) = \frac{d}{dz}(p(z) + q(z)) \\
 = \frac{d}{dz}p(z) + \frac{d}{dz}q(z) \\
= D(p(z)) + D(q(z)),
\]
and if \( \lambda \in \mathbb{R} \) then
\[
 D(\lambda p(z)) = \frac{d}{dz}(\lambda p(z)) \\
= \lambda \frac{d}{dz}p(z) \\
= \lambda D(z).
\]
(b) Let $v_1 = 1$, $v_2 = z$, $v_3 = z^2$, $v_4 = z^3$. Then
\[
[D]_{B \rightarrow B} = ([D(v_1)]_{B}[D(v_2)]_{B}[D(v_3)]_{B}[D(v_4)]_{B})
= ([0]_{B}[1]_{B}[2z]_{B}[3z^2]_{B})
= \begin{pmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 2 & 0 \\
0 & 0 & 0 & 3 \\
0 & 0 & 0 & 0
\end{pmatrix}.
\]

(c) We have
\[
[D]_{C \rightarrow B} = ([D(v_1)]_{C}[D(v_2)]_{C}[D(v_3)]_{C}[D(v_4)]_{C})
= ([0]_{C}[1]_{C}[2z]_{C}[3z^2]_{C})
= \begin{pmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0
\end{pmatrix}.
\]