Notes on Homework 7

1. (a) Two things to show:

**Additivity** Suppose \( \vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \vec{y} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \in \mathbb{R}^3 \). Then

\[
f(\vec{x} + \vec{y}) = f\left(\begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \\ x_3 + y_3 \end{pmatrix}\right)
= \begin{pmatrix} 2(x_1 + y_1) + (x_2 + y_2) \\ 4(x_3 + y_3) \end{pmatrix}
= \begin{pmatrix} 2x_1 + x_2 \\ 4x_3 \end{pmatrix} + \begin{pmatrix} 2y_1 + y_2 \\ 4y_3 \end{pmatrix}
= f(\vec{x}) + f(\vec{y}).
\]

In fact, this isn’t the way one would probably derive this solution; in practice, the way to solve this part is probably to compute \( f(\vec{x} + \vec{y}) \) and \( f(\vec{x}) + f(\vec{y}) \) separately, and then observe that they’re the same vector.

**Homogeneity** Suppose \( \vec{x} \in \mathbb{R}^3, \lambda \in \mathbb{R} \). Then

\[
f(\lambda \vec{x}) = f\left(\begin{pmatrix} \lambda x_1 \\ \lambda x_2 \\ \lambda x_3 \end{pmatrix}\right)
= \begin{pmatrix} 2(\lambda x_1 + \lambda x_2) \\ 4\lambda x_3 \end{pmatrix}
= \lambda \begin{pmatrix} 2x_1 + x_2 \\ 4x_3 \end{pmatrix}
= \lambda f(\vec{x}).
\]

See remarks above.

(b) Fix a matrix \( B \). Two things to show:

**Additivity** Suppose \( A, C \in \text{Mat}_{n,n}(\mathbb{F}) \). Note that \( B \) is a fixed matrix throughout this problem, so we’re not allowed to modify it. Then

\[
g(A) + g(C) = AB - BA + CB - BC
\]
while

\[ g(A + C) = (A + C)B - B(A + C) \]
\[ = AB + CB - BA - BC \text{ start using matrix properties} \]
\[ = AB - BA + CB - BC \]
\[ = g(A) + g(C) \text{ from the first line.} \]

**homogeneity** Suppose \( A \in \text{Mat}_{n \times n}(\mathbb{F}) \) and \( \lambda \in \mathbb{F} \). Using properties of matrix multiplication from about a month ago, we have

\[ g(\lambda A) = \lambda AB - B\lambda A \]
\[ = \lambda AB - \lambda BA \]
\[ = \lambda(AB - BA) \]
\[ = \lambda g(A) \]

(c) Same story:

**additivity** Suppose \( p(z) = az^2 + bz + c, \ q(z) = dz^2 + ez + f, \ p(z), q(z) \in \mathcal{P}_2(\mathbb{R})[z]. \)

Then

\[ h(p(z) + q(z)) = h((a + d)z^2 + (b + e)z + (c + f)) \]
\[ = \begin{pmatrix} (a + d) + (b + e) \\ (b + e) - (c + f) \\ 2(c + f) \end{pmatrix} \]

while

\[ h(p(z)) + h(q(z)) = \begin{pmatrix} a + b \\ b - c \\ 2c \end{pmatrix} + \begin{pmatrix} d + e \\ e - f \\ 2f \end{pmatrix} \]
\[ = \begin{pmatrix} (a + d) + (b + e) \\ (b + e) - (c + f) \\ 2(c + f) \end{pmatrix}. \]
\textbf{homogeneity} Suppose $\lambda \in \mathbb{R}$, $p(z) = az^2 + bz + c \in \mathcal{P}_2(\mathbb{R})[z]$. Then
\[
h(\lambda p(z)) = h(\lambda (az^2 + bz + c)) \\
= h(\lambda az^2 + \lambda bz + \lambda c) \\
= \begin{pmatrix} \lambda a + \lambda b \\ \lambda b - \lambda c \\ 2\lambda c \end{pmatrix} \\
= \lambda \begin{pmatrix} a + b \\ b - c \\ 2c \end{pmatrix} \\
= \lambda h(p(z)).
\]

2. (a) By definition, we have
\[
T(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}) = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\
= \begin{pmatrix} x_1 + 3x_2 \\ 2x_1 + 4x_2 \end{pmatrix}.
\]
(b) Our formula yields
\[
\begin{pmatrix} 3 + 3 \cdot 1 \\ 2 \cdot 3 + 4 \end{pmatrix} = \begin{pmatrix} 6 \\ 10 \end{pmatrix},
\]
while the definition yields
\[
T(\begin{pmatrix} 3 \\ 1 \end{pmatrix}) = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 1 \end{pmatrix} \\
= \begin{pmatrix} 6 \\ 10 \end{pmatrix}.
\]

3. (a) Note that
\[
f(\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}) = \begin{pmatrix} x_1 - x_2 \\ x_1 + 4x_3 \\ -x_1 \end{pmatrix} \\
= x_1 \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} + x_2 \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 0 \\ 4 \\ 0 \end{pmatrix}.
\]
So in fact, we can compute $f(\vec{x})$ via
\[
f(\vec{x}) = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 0 & 4 \\ -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.
\]
(b) Compute in two different ways; using the given formula,

\[
T\left(\begin{pmatrix} 1 \\
3 \\
-1 \end{pmatrix}\right) = \begin{pmatrix} 1 - 3 \\
1 + 4(-1) \\
-1 \end{pmatrix}
\]
\[
= \begin{pmatrix} -2 \\
-3 \\
-1 \end{pmatrix}
\]

while multiplying by our matrix yields

\[
\begin{pmatrix} 1 & -1 & 0 \\
1 & 0 & 4 \\
-1 & 0 & 0 \end{pmatrix}
\begin{pmatrix} 1 \\
3 \\
-1 \end{pmatrix} = \begin{pmatrix} -2 \\
-3 \\
-1 \end{pmatrix}.
\]

4. (a) The reduced row echelon form of \( A \) is

\[
\begin{pmatrix} 1 & 0 & -\frac{17}{2} & -\frac{19}{2} \\
0 & 1 & \frac{9}{2} & \frac{11}{2} \end{pmatrix}.
\]

(Computation omitted here.) Therefore, the most general solution to the equation \( Ax = 0 \) is

\[
\begin{pmatrix} \frac{17}{2}x_3 + \frac{19}{2}x_4 \\
-\frac{9}{2}x_3 - \frac{11}{2}x_4 \\
x_3 \\
x_4 \end{pmatrix}
\]

and the null space of \( A \) is

\[
\text{null}(A) = \text{span}\left(\begin{pmatrix} \frac{17}{2} \\
\frac{9}{2} \\
1 \\
0 \end{pmatrix}, \begin{pmatrix} \frac{19}{2} \\
-\frac{11}{2} \\
0 \\
1 \end{pmatrix}\right).
\]

These two vectors are linearly independent check this, and \( \dim \ker(A) = 2 \). A two-dimensional subspace maps to zero, and \( f \) is not injective.

Remark: Actually, you can figure out that \( f \) is not injective without computing anything! Since
\( \dim \mathbb{R}^4 = 4 > \dim \mathbb{R}^2 = 2 \), there is no injective linear transformation \( \mathbb{R}^4 \to \mathbb{R}^2 \).

(b) Since the reduced row echelon form of \( B \) is

\[
\begin{pmatrix} 1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \end{pmatrix}
\]

(calculation omitted), for any \( b \in \mathbb{R}^3 \) there is a unique \( x \in \mathbb{R}^3 \) such that \( Ax = b \). In particular, the equation \( Ax = 0 \) has a unique solution, and \( g \) is injective.
5. The key observation for this problem is that for any vector \( b = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \), the equation \( Ax = b \) is consistent. Moreover, each variable is a pivot variable, so there is exactly one solution to \( Ax = b \) for any particular \( b \). (Explicitly, the solution is \( x_3 = b_3; x_2 = b_2 - 2x_3 = b_2 - 2b_3 \), and \( x_1 = b_1 - 2x_2 - 3x_3 = b_1 - 2b_2 + b_3 \).)

(a) Since \( Ax = b \) always has a solution, each element \( b \in \mathbb{R}^3 \) is in the column space of \( A \), and the associated linear transformation \( T \) is surjective.

(b) Since \( T \) is surjective, we know that \( T \) is bijective if and only if it is injective, too. From class, we know that \( T \) is injective if and only if the nullspace of \( A \) is trivial, that is, if and only if the solution to \( T(\vec{x}) = \vec{0} \) is the zero vector. This last condition is true; we can read this off from the echelon form, as discussed above. Alternatively, since

\[
\dim \text{im}(T) + \dim \ker(T) = \dim(\text{source}) = 3,
\]

and since \( \dim \text{im}(T) = 3 \), the kernel has dimension zero, and thus \( T \) is injective.