Notes on Homework 4

1. A set of vectors is linearly independent if the only linear combination of them which yields \( \vec{0} \) is the trivial linear combination (i.e., if each coefficient is zero). So, let’s see what happens:

\[
\alpha_1 \cdot \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} + \alpha_2 \cdot \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} + \alpha_3 \cdot \begin{pmatrix} 1 \\ 0 \\ 6 \end{pmatrix} + \alpha_4 \cdot \begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.
\]

This gives us a linear system:

\[
\begin{align*}
\alpha_1 + 2\alpha_2 + \alpha_3 + 4\alpha_4 &= 0 \\
3\alpha_1 - \alpha_2 + \alpha_4 &= 0 \\
6\alpha_1 + \alpha_2 + 6\alpha_3 + 2\alpha_4 &= 0 \\
2\alpha_1 + 4\alpha_2 + 10\alpha_4 &= 0
\end{align*}
\]

By moving to the augmented matrix and putting that into RREF, we actually find infinitely many solutions (one of which is \((-1, -2, 1, 1)\)). Thus, there is (at least!) one nontrivial linear combination of the vectors yielding \( \vec{0} \). Thus, they are linearly dependent.

2. Try the linear combination \( 1 \cdot \vec{0} + 0 \cdot \vec{v} \). It is nontrivial (not all coefficients are 0), and it certainly yields \( \vec{0} \).

3. Let’s expand along the bottom row (since it has the most zeros):

\[
\det \begin{pmatrix} 1 & 1 & 0 & 3 \\ 2 & 0 & 1 & 2 \\ 1 & 1 & 2 & 1 \\ 0 & 1 & 0 & 1 \end{pmatrix} = -0 \cdot \det \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 1 & 2 & 1 \end{pmatrix} + 1 \cdot \det \begin{pmatrix} 1 & 0 & 3 \\ 2 & 1 & 2 \\ 1 & 1 & 1 \end{pmatrix} - 0 \cdot \det \begin{pmatrix} 1 & 1 & 3 \\ 2 & 0 & 2 \\ 1 & 1 & 1 \end{pmatrix} + 1 \cdot \det \begin{pmatrix} 1 & 1 & 0 \\ 2 & 0 & 1 \\ 1 & 1 & 2 \end{pmatrix} = 0 + 1 \cdot (-3 + 9) + 0 + 1 \cdot (-1 - 3) = 2.
\]

4. (a) \( A^{-1} = \begin{pmatrix} 3/11 & -1/4 \\ 1/11 & 3/11 \end{pmatrix} \), \( \det(A) = 11 \).

(b) The determinant is 0, and the matrix has no inverse. Interesting....

5. (a) For \( k = 2 \), there are clearly three flops involved with computing \( a \cdot d - b \cdot c \). That is our base case.

In general, we’re trying to express the number of flops it takes to compute the determinant of a \( k \times k \) matrix, using only the definition of the determinant. Since this involves
taking the determinant of several \((k - 1) \times (k - 1)\)-matrices, the answer will be expressed in terms of \(N_{k-1}\), the number of flops necessary to compute such determinants. Let’s expand along the first row of our \(k \times k\) matrix. At the first entry, we compute the determinant of a \((k - 1) \times (k - 1)\) matrix (requiring \(N_{k-1}\) flops, by definition) and multiply that by the \((1,1)\) entry of our matrix (one more flop). Thus, for that entry, we have \(N_{k-1} + 1\) flops. There are \(k\) entries in the first row of our matrix, so we have \(k \cdot (N_{k-1} + 1)\) flops. All that remains is to add up those quantities based at each entry. There are \(k\) of those quantities, requiring \(k - 1\) more flops. That brings our total up to \(N_k = k \cdot (N_{k-1} + 1) + k - 1 = k \cdot (N_{k-1} + 2) - 1\) flops, as desired.

\[(b) \quad N_2 = 3, \quad N_3 = (N_2 + 2) \cdot 3 - 1 = 14, \quad N_4 = (N_3 + 2) \cdot 4 - 1 = 63, \quad N_5 = 324, \quad N_6 = 1955, \quad N_7 = 13698, \quad N_8 = 109599, \quad N_9 = 986408, \quad N_{10} = 9864099.\] A 10 \(\times\) 10 matrix isn’t very big (especially in applications), but computing the determinant of such a matrix in this way (which has only 100 entries) requires nearly 10 million operations!!