Notes on Homework 3

1. (a) \[ \text{span} \left\{ \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\} = \left\{ \alpha \cdot \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + \beta \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} : \alpha, \beta \in \mathbb{R} \right\} \]

\[ = \left\{ \begin{pmatrix} 0 \\ \alpha + \beta \\ \alpha \end{pmatrix} : \alpha, \beta \in \mathbb{R} \right\}. \]

(b) and (c) No sketch here, but the first vector gives some vector “between” the y– and z–axes while the second gives just the y-axis. The span is then the entire \((y, z)\)–plane.

2. (a) \[ \text{span} \left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix} \right\} = \left\{ \alpha \cdot \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \beta \cdot \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix} : \alpha, \beta \in \mathbb{R} \right\} \]

\[ = \left\{ \begin{pmatrix} \alpha + 2\beta \\ 2\alpha + 4\beta \\ 3\alpha + 6\beta \end{pmatrix} : \alpha, \beta \in \mathbb{R} \right\}. \]

This is technically correct, even though it is not the simplest representation. There are two ways to see the simpler representation. First, you could notice that all three entries are multiples of \(\alpha + 2\beta\) and rename that (say, \(\gamma = \alpha + 2\beta\)). Then, the representation is

\[ \left\{ \begin{pmatrix} \gamma \\ 2\gamma \\ 3\gamma \end{pmatrix} : \alpha, \beta \in \mathbb{R} \right\}. \]

Alternatively, you could observe that \(\begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix}\) is a multiple of \(\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}\). Thus, the former is in the span of the latter (or vice versa), so you can ignore one of them.

(b) and (c) Again, no sketch here. One vector, by itself, spans a line (linear combinations of one vectors are just multiples of that vector). Since the two vectors of this problem sit on the same line, their span is still just the line through \((1, 2, 3)\).

3. Suppose \(u \in \text{span}(v_1, \cdots, v_r)\). By definition, this means that there are numbers \(a_1, \cdots, a_r\) such that \(u = \sum_{i=1}^{r} a_i v_i\). Then

\[ \lambda u = \lambda \sum_{i=1}^{r} a_i v_i \]

\[ = \sum_{i=1}^{r} \lambda(a_i v_i) \text{, distributive property} \]

\[ = \sum_{i=1}^{r} (\lambda a_i) v_i \text{, associativity of scalar multiplication} \]

\[ \in \text{span}(v_1, \cdots, v_r), \]
since we have expressed $\lambda u$ as a linear combination of the vectors $v_1, \cdots, v_r$.

4. We want to try to write
   \[
   \begin{pmatrix}
   3 \\
   8
   \end{pmatrix} = \alpha \cdot \begin{pmatrix}
   2 \\
   1
   \end{pmatrix} + \beta \cdot \begin{pmatrix}
   1 \\
   3
   \end{pmatrix},
   \]
   for some $\alpha, \beta \in \mathbb{R}$.

   This conveniently provides us with a linear system:
   \[
   \begin{align*}
   2\alpha + \beta &= 3 \\
   \alpha + 3\beta &= 8
   \end{align*}
   \]

   We know how to solve these sorts of systems, and we find that $\alpha = \frac{1}{5}, \beta = \frac{13}{5}$. This is easily checked:
   \[
   \frac{1}{5} \begin{pmatrix}
   2 \\
   1
   \end{pmatrix} + \frac{13}{5} \begin{pmatrix}
   1 \\
   3
   \end{pmatrix} = \begin{pmatrix}
   3 \\
   8
   \end{pmatrix}. \checkmark
   \]

5. (a) It is no problem to read the general solution from the coefficient matrix (already in RREF) and constant of vectors. From that, it is easy to see that one solution is
   \[
   \begin{pmatrix}
   1 \\
   0 \\
   5 \\
   0
   \end{pmatrix}.
   \]

   (Other solutions are fine, too).

   (b) \[
   \begin{pmatrix}
   1 \\
   0 \\
   5 \\
   0
   \end{pmatrix} + x_2 \cdot \begin{pmatrix}
   -4 \\
   1 \\
   0 \\
   0
   \end{pmatrix} + x_4 \cdot \begin{pmatrix}
   -3 \\
   0 \\
   2 \\
   1
   \end{pmatrix}.
   \]

   (c) The vectors associated with the free variables are typically called “basic solutions” (or BSAHS, if you took 229). If you change the vector of constants in the problem definition, these will not change (they solve the homogeneous system). All that changes is this extra vector which has no associated free variable, the so-called “particular” or “distinguished” solution.

   (d) In light of the previous part, only the “distinguished” solution changes. In this case, it will be $\vec{0}$.

6. (a) There are infinitely many answers here. Two vectors that lie on the same line won’t work. Otherwise, any two nonzero vectors would work. The canonical choice is the set of vectors \[
   \left\{ \begin{pmatrix}
   1 \\
   0
   \end{pmatrix}, \begin{pmatrix}
   0 \\
   1
   \end{pmatrix} \right\}.
   \]

   (b) Again, there are infinitely many choices here. Take your answer to part (a) and toss in any other 2-vectors that you want. The point here is that the two vectors in part (a) already span all of $\mathbb{R}^2$ (the (x,y)-plane, if you like). Thus, throwing in more vectors doesn’t increase the span at all.
(c) Two. Zero vectors spans nothing. One vector spans a line (as discussed above). Two vectors will obviously be enough (assuming they are not chosen in some degenerate way) since you answered part (a). The point here is that you need at least two vectors to span two-dimensional space, and two vectors (same caveat as in the last sentence) will suffice.