Notes on Homework 1

These are just the final answers, sometimes with some intermediate steps. To receive full credit, all work must be shown.

1. Final answer is \( x = 2, \ y = -2 \). There should be graph of, for instance, the lines \( x + 2y = -2; \ 2x - y = 6; \) and something like, say, \( -5y = 10 \).

2. Set up a system of equations, such as:
\[
5c + 2s = 10 \\
2c + 5s = 5;
\]
final answer is \( c = 40/21, \ s = 5/21 \).

3. (a) Let \( S \) be the set of solutions to the first system of equations, and let \( T \) be the set of solutions to the second system. We’ll show that \( S = T \), i.e., that \( (x_1 = \beta_1, \cdots, x_n = \beta_n) \) is a solution to the first system if and only if it’s a solution to the second system.

\( S \subseteq T \) Suppose \( (\beta_1, \cdots, \beta_n) \in S \). This means that for each \( i \) with \( 1 \leq i \leq m \), we have \( f_i(\beta_1, \cdots, \beta_n) = b_i \).

Consider the \( i^{th} \) equation of the second system. If \( i \geq 2 \), then (again) \( f_i(\beta_1, \cdots, \beta_n) = b_i \), so that \( (\beta_1, \cdots, \beta_m) \) solves the \( i^{th} \) equation. For \( i = 1 \), we have
\[
\lambda f_1(\beta_1, \cdots, \beta_n) = \lambda \cdot b_1
\]
since \( (\beta_1, \cdots, \beta_n) \) solves the first equation of the first set.

\( T \subseteq S \) Conversely, suppose \( (\beta_1, \cdots, \beta_n) \) solves the second system of equations. As before, equations \( 2, \cdots, m \) are automatically satisfied. To verify the first equality, compute:
\[
f_1(\beta_1, \cdots, \beta_n) = \frac{1}{\lambda} \cdot \lambda f_1(\beta_1, \cdots, \beta_n) \\
= \frac{1}{\lambda} \lambda b_1 \text{ since } \beta \text{ solves } \lambda f_1 = \lambda b_1 \\
= b_1
\]
as desired.

(b) We can’t divide by \( \lambda \). Optional: Concretely, as an example consider the solution set to
\[
x_1 + x_2 = 0 \\
x_1 - x_2 = 0
\]
The only solution to this system is \( x_1 = x_2 = 0 \). If we multiply the first equation by zero then there’s no constraint at all, and any pair of numbers of the form \( x_1 = t, x_2 = t \) will do.
4. Some of the steps:

(a) Start with the system

\[
\begin{align*}
x + 2y - 2z &= -3 \\
-2x - 3y + 3z &= 3 \\
2x + 6y - 5z &= -3
\end{align*}
\]

I’m not going to describe all the steps, just the outcome:

\[
\begin{align*}
x + 2y - 2z &= -3 \\
y - z &= -3 \\
2y - z &= 3
\end{align*}
\]

and then

\[
\begin{align*}
x + 2y - 2z &= -3 \\
y - z &= -3 \\
z &= 9
\end{align*}
\]

Then \(y = -3 + z = 6\), and \(x = -3 - 2y + 2z = 3\).
So the only answer is \((x, y, z) = (3, 6, 9)\).

(b)

\[
\begin{pmatrix}
1 & 2 & -2 & -3 \\
-2 & -3 & 3 & 3 \\
2 & 6 & -5 & -3
\end{pmatrix}
\]

(c)

\[
\begin{pmatrix}
1 & 0 & 0 & 3 \\
0 & 1 & 0 & 6 \\
0 & 0 & 1 & 9
\end{pmatrix}
\]

5. (a) Some steps along the way:

\[
\begin{align*}
w + 2x + y + 9z &= 2 \\
-w - x - 4z &= -3 \\
w + x + y + 8z &= 1
\end{align*}
\]

Then

\[
\begin{align*}
w + 2x + y + 9z &= 2 \\
x + y + 5z &= -1 \\
-x - z &= -1
\end{align*}
\]
and then

\[ \begin{align*}
   w + 2x + y + 9z &= 2 \\
   x + z &= 1 \\
   y + 4z &= -2
\end{align*} \]

Thus, for any value of \( z \), \((w, x, y) = (1 - z, -2 - 4z, 2 - 3z)\) is a solution; and these are all of them.

(b)

\[
\begin{pmatrix}
  1 & 2 & 1 & 9 & 2 \\
  -1 & -1 & 0 & -4 & -3 \\
  1 & 1 & 1 & 8 & 1
\end{pmatrix}
\]

(c)

\[
\begin{pmatrix}
  1 & 0 & 0 & 3 & 2 \\
  0 & 1 & 0 & 1 & 1 \\
  0 & 0 & 1 & 4 & -2
\end{pmatrix}
\]

(d) (Same as before)

6. (a) We have the system

\[ \begin{align*}
   x + y + 5x &= 1 \\
   -2x - y - 8z &= -3 \\
   x - y + z &= 4
\end{align*} \]

Then

\[ \begin{align*}
   x + y + 5x &= 1 \\
   y + 2z &= 1 \\
   -2y - 4z &= 2
\end{align*} \]

\[ \begin{align*}
   x + y + 5x &= 1 \\
   y + 2z &= 1 \\
   y + 2z &= -1
\end{align*} \]

But these last two equations are inconsistent, and thus there are no solutions to this system.

(b)

\[
\begin{pmatrix}
  1 & 1 & 5 & 1 \\
  0 & 1 & 2 & 1 \\
  0 & 0 & 0 & 4
\end{pmatrix}
\]
(c) From the shape of the last row, we see that there are no solutions.

7. (a) Yes, it’s in RREF.
   (b) No, the fourth column is a pivot column, but the leading entry in row three is not a one, and it’s not the only nonzero entry in its column.
   (c) No, the pivots are 2, instead of 1.
   (d) Yes
   (e) No, the first nonzero entry in row one comes after the first nonzero entry in row two.