Homework 8
Due: Wednesday, October 29

1. Consider the following vectors in \( \mathbb{R}^2 \):

\[
x = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad y = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad z = \begin{pmatrix} 3 \\ 4 \end{pmatrix}.
\]

In each case, is there a linear transformation \( \mathbb{R}^2 \to \mathbb{R}^2 \) which maps the vectors \( x, y \) and \( z \) as prescribed? Explain. (Hint: \( \{ x, y \} \) is a basis for \( \mathbb{R}^2 \).)

(a) \( x \mapsto (2, 3)^T; y \mapsto (0, 1)^T; z \mapsto (1, 5)^T. \)

(b) \( x \mapsto (2, 3)^T; y \mapsto (0, 1)^T; z \mapsto (2, 5)^T. \)

(c) \( x \mapsto (1, 1)^T; y \mapsto (0, 1)^T; z \mapsto (3, 3)^T. \)

2. Suppose that \( f \in L(W, V) \) is surjective and that \( w_1, \ldots, w_n \) spans \( W \). Prove that \( f(w_1), \ldots, f(w_n) \) spans \( V \).

3. Let \( A \in \text{Mat}(m, n, \mathbb{F}) \) be a matrix. For each of the following statements, indicate whether the statement is true or false. (For once, you need not justify your answer!)

   (a) If \( \text{rank}(A) = m \) then the system \( Ax = b \) has at least one solution for every \( b \in \mathbb{F}^m \).

   (b) If \( \text{rank}(A) = n \) then the system \( Ax = b \) has at least one solution for every \( b \in \mathbb{F}^m \).

   (c) If \( \text{rank}(A) = m \) then the system \( Ax = b \) has at most one (i.e., no solution or a unique solution) for every \( b \in \mathbb{F}^m \).

   (d) If \( \text{rank}(A) = n \) then the system \( Ax = b \) has at most one (i.e., no solution or a unique solution) for every \( b \in \mathbb{F}^m \).

   (e) If \( Ax = b \), then \( b \in \text{col}(A) \).

   (f) If \( B \in \text{Mat}(m, n, \mathbb{F}) \) is row equivalent to \( A \) then \( \text{col}(A) = \text{col}(B) \).

4. Let \( V = \mathbb{R}^2 \) with the standard basis \( \mathcal{E} = \{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \} \). Let \( f : V \to V \) be the linear transformation with matrix

\[
[f]_{\mathcal{E}-\mathcal{E}} = \begin{pmatrix} -3 & -8 \\ 10 & 27 \end{pmatrix}.
\]

(a) Suppose \( [v_1]_\mathcal{E} = \begin{pmatrix} 5 \\ 3 \end{pmatrix} \). What is \( [f(v_1)]_\mathcal{E} \)?

(b) Let \( \mathcal{C} \) be the basis \( \mathcal{C} = \{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \end{pmatrix} \} \). What is \( [f]_{\mathcal{C}-\mathcal{C}} \)?

(c) Suppose \( [v_2]_\mathcal{C} = \begin{pmatrix} 3 \\ 4 \end{pmatrix} \). What is \( [f(v_2)]_\mathcal{C} \)?
5. Let $V = \mathbb{R}^2$ with the standard basis $\mathcal{E} = \{(1,0), (0,1)\}$, and consider the following pictures:

(a) A linear transformation $f_{21} : V \rightarrow V$ transforms picture (1) to picture (2). What is $[f_{21}]_{\mathcal{E}\rightarrow\mathcal{E}}$?

(b) Do the same thing for the rest of the pictures; for each $j = 2, \cdots, 7$, find the matrix of the linear transformation $f_{j1}$ which transforms picture (1) to picture (j).

6. Consider the matrix

$$A = \begin{pmatrix} -12 & 30 \\ -5 & 13 \end{pmatrix}.$$ 

(a) Confirm that $A$ has eigenvectors $(3,1)^T$ and $(2,1)^T$.

(b) Find the eigenvalues corresponding to each eigenvector.

(c) What are the geometric and algebraic multiplicities of these eigenvalues?

(d) Write down one vector from each of the eigenspaces of $A$ other than the vectors in part (a).

(HINT: This problem is just to make sure you know the definitions – there is nothing tricky about it.)