Homework 7
Due: Wednesday, October 22

1. For this problem, you will be asked to prove that three different functions between finite-dimensional vector spaces are linear transformations. In particular, you will need to show in each case that the function respects both addition and scalar multiplication. In other words, for \( f : V \rightarrow W \), show that

(i) \( f(v_1 + v_2) = f(v_1) + f(v_2) \) for \( v_1, v_2 \in V \) and

(ii) \( f(\lambda v) = \lambda f(v) \) for \( \lambda \in \mathbb{F}, v \in V \). The notation is a bear, but when you get past that, these actually aren’t so bad....

(a) Consider the function

\[
\begin{pmatrix}
\mathbb{R}^3 \\
\mathbb{R}^2 \\
\end{pmatrix} \rightarrow
\begin{pmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
\end{bmatrix} \\
\begin{bmatrix}
2x_1 + x_2 \\
4x_3 \\
\end{bmatrix} \\
\end{pmatrix}
\]

Show that this map is a linear transformation.

(b) Let \( B \in \text{Mat}_{n,n}(\mathbb{F}) \) (the set of all \( n \times n \) matrices with entries in the field \( \mathbb{F} \)) be some fixed matrix. Consider the function

\[
\begin{pmatrix}
\text{Mat}_{n,n}(\mathbb{F}) \\
\text{Mat}_{n,n}(\mathbb{F}) \\
\end{pmatrix} \rightarrow
\begin{pmatrix}
A \\
A \cdot B - B \cdot A \\
\end{pmatrix}
\]

In other words, this function takes in the matrix \( A \) and outputs the matrix \( A \cdot B - B \cdot A \). Show that this is a linear transformation.

(c) Consider the function

\[
\begin{pmatrix}
\mathbb{P}_2(\mathbb{R})[z] \\
\mathbb{R}^3 \\
\end{pmatrix} \rightarrow
\begin{pmatrix}
az^2 + bz + c \\
\begin{bmatrix}
a + b \\
b - c \\
2c \\
\end{bmatrix} \\
\end{pmatrix}
\]

Show that this function is a linear transformation.

2. Given a matrix \( A \in \text{Mat}_{m,n}(\mathbb{F}) \), there is a corresponding linear transformation \( T : \mathbb{R}^n \rightarrow \mathbb{R}^m \) given by \( T(x) = A \cdot x \):

(a) The matrix \( A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \) can be used to define a linear transformation \( T : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) via left multiplication. What is \( T( \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} ) \)? (HINT: Compute \( A \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \).)
(b) For \( x = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \), compute both \( T(x) \) and \( A \cdot x \). (HINT: They should be the same!)

3. Conversely, a linear transformation can be expressed by a matrix.

(a) Consider the linear transformation

\[
\mathbb{R}^3 \xrightarrow{f} \mathbb{R}^3
\]

\[
\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \mapsto \begin{bmatrix} x_1 - x_2 \\ x_1 + 4x_3 \\ -x_1 \end{bmatrix}
\]

Write down a matrix \( A \) such that \( T(x) = A \cdot x \).

(b) For \( x = \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix} \), compute both \( T(x) \) and \( A \cdot x \). (HINT: They should be the same!)

4. Recall that the nullspace of a linear transformation is the set of all vectors that get mapped to \( \vec{0} \). Thinking of a linear transformation as a matrix \( A \), \( x \) is in the nullspace if \( Ax = 0 \).

(a) Compute the nullspace of \( f \) if \( f : \mathbb{R}^4 \to \mathbb{R}^2 \) is given by \( f(x) = Ax \) where

\[
A = \begin{bmatrix} 1 & 3 & 5 & 7 \\ 2 & 4 & 1 & 3 \end{bmatrix}
\]

Is \( f \) injective? (HINT: You should be able to express the nullspace as the span of two vectors.)

(b) Compute the nullspace of \( g \) if \( g : \mathbb{R}^3 \to \mathbb{R}^3 \) is given by \( g(x) = Bx \) where

\[
B = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 1 \\ 1 & 1 & 1 \end{bmatrix}
\]

Is \( g \) injective?

5. Recall that the matrix-vector product \( Ax \) can be written \( A_1x_1 + A_2x_2 + \cdots + A_nx_n \) where \( A_i \) is just the \( i^{th} \) column of \( A \). Thus, the columns of \( A \) span the range of the linear transformation \( T \) given by \( T(x) = A \cdot x \).

(a) Consider the matrix \( A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \). Is the corresponding linear transformation \( T(x) = A \cdot x \) surjective?

(b) Compute the nullspace of \( T \) (as in problem 4). Is \( T \) bijective?