Homework 10/Practice Test 2
I WILL NOT BE COLLECTING THIS HOMEWORK ASSIGNMENT!

Remark  The actual test will be a shortened, tweaked version of this practice test, so know this stuff (and closely related concepts) well....

1. Briefly (but carefully) define the following terms:
   (a) Subspace
   (b) Linear transformation
   (c) Eigenvalue
   (d) Image (of a linear transformation)
   (e) Injective linear transformation
   (f) Spanning set
   (g) Basis

2. State the three criteria for determining whether a subset of a vector space is a subspace.

3. State the two criteria for determining whether a function between two vector spaces is a linear transformation.

4. $A$ is singular. Tell me 8 other (distinct) things that you know about $A$.

5. Let $V = \mathcal{P}(\mathbb{F})[z]$, the vector space of all polynomials over some field $\mathbb{F}$. Show that the subset $S = \{p(z) \in V | p(0) = 0\}$, i.e., the set of all polynomials with 0 constant term, is a subspace of $V$.

6. Consider the function

$$
T : \mathbb{R}^3 \xrightarrow{f} \mathbb{R}^2
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
\end{bmatrix}
\mapsto
\begin{bmatrix}
x_1 - x_2 \\
2x_3 + x_2
\end{bmatrix}.
$$

(a) Show that $T$ is a linear transformation.
(b) Write down the matrix for this linear transformation (using the standard bases on $\mathbb{R}^3$ and $\mathbb{R}^2$.)
(c) Is this transformation injective?

7. Consider the matrix

$$
A = \begin{pmatrix} 1 & 5 \\ 5 & 1 \end{pmatrix}
$$
(a) Find the characteristic polynomial of $A$.
(b) Find all eigenvalues of $A$.
(c) Find all eigenvectors of $A$.
(d) Without attempting to diagonalize it, is $A$ diagonalizable? Why or why not?
(e) Without attempting to invert it, is $A$ invertible? Why or why not?

8. Let $A$ be as in the last problem.
   (a) Suppose that $A = [f]_{\varepsilon \leftarrow \varepsilon}$, where $\varepsilon$ is the standard basis for $\mathbb{R}^2$.
   (b) Let $B = \left\{ \begin{pmatrix} 1 \\ 2 \\ \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \end{pmatrix} \right\}$ be another basis for $\mathbb{R}^2$. Write down $[id]_{B \leftarrow \varepsilon}$ and $[id]_{\varepsilon \leftarrow B}$.
   (c) Using these, what is $[f]_{B \leftarrow B}$?

9. For each of the following statements, indicate whether the statement is true or false, and give a short (few sentence) justification.
   (a) If $f \in \mathcal{L}(W, V)$ and $\dim W > \dim V$, then $f$ is not injective.
   (b) If $v_1, \ldots, v_{n+1}$ are elements of $V$, and if the set $\{v_1, \ldots, v_n\}$ is linearly independent, then $\{v_1, \ldots, v_n, v_{n+1}\}$ is linearly independent.
   (c) If $v_1, \ldots, v_{n+1}$ are elements of $V$, and if the set $\{v_1, \ldots, v_n\}$ spans $V$, then $\{v_1, \ldots, v_n, v_{n+1}\}$ spans $V$.
   (d) If $V$ is a vector space and if $U \subset V$ and $W \subset V$ are subspaces, then $U \cup W$ is a subspace of $V$. 