Math 369 Practice Final

The final won’t be this long. The point of this practice test is to get you thinking about the main points of the course and revisiting some of the older concepts, proof techniques, and computations. The topics considered below aren’t exhaustive (considering everything we have talked about), but they should get you in pretty good shape for the final. Check out your old homework sets and exams, too.

1. Describe the solution set of the following system of linear equations
   
   \[ 2x + 4y = 8 \]
   \[ x - 3y = 2. \]

   (a) Think about how to recognize the cases of unique solutions, no solutions, and infinitely many solutions.

   (b) How do you write down the solutions if there are infinitely many of them?

   (c) How can you check your solutions?

2. What does it mean to be a subspace of a finite-dimensional vector space? How would you prove that some subset is a subspace?

3. What is a basis of a vector space?

4. What is dimension? What does it have to do with bases?

5. What does it mean to say that some set of vectors is linearly independent? How can you prove that some set of vectors is linearly dependent?

6. Must bases be linearly independent? Must spanning sets be linearly independent?

7. Suppose \( V = \{v_1, \ldots, v_N\} \) is a set of linearly independent vectors.

   (a) Is \( \{v_1, \ldots, v_{N-1}\} \) linearly independent?

   (b) If I extend \( V \) by tossing in some other vector, will this new set be linearly independent?

   (c) What if I throw in the zero vector?

8. What is a linear transformation between two vector spaces? How can you check that something is a linear transformation?

9. Let \( T : W \to V \) be some linear transformation between two vector spaces with \( \dim(W) = n \) and \( \dim(V) = m. \)

   (a) What are the kernel and image of \( T \) (the definition, that is)? Are they always subspaces?

   (b) Define rank and nullity. What does the rank-nullity theorem say? (Double check: Is it \( m \) or \( n \)?)
(c) What does it mean to say that $T$ is injective, surjective, or bijective?

10. Find the inverse of $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 6 & 8 \\ 1 & -1 & 0 \end{pmatrix}$.

(a) How could you recognize that a matrix is not invertible?
(b) Remember those long lists of properties equivalent to singularity/nonsingularity? Check those out again. We won’t ask for them explicitly, but they may be handy.

11. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation given by $T([v_1, v_2]^T) = [2v_1 + 3v_2, 5v_1]^T$.

(a) First, check that this is indeed a linear transformation.
(b) What is $[T]_{\varepsilon \leftarrow \varepsilon}$, i.e., the matrix associated to this transformation using the standard basis for $\mathbb{R}^2$?
(c) What is the change of basis matrix $[id]_{B \leftarrow \varepsilon}$, from $\varepsilon = \left\{ \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \end{pmatrix} \right\}$?
(d) What is the change of basis matrix $[id]_{\varepsilon \leftarrow B}$? (HINT: This is really easy since we already know the basis elements of $B$ in terms of the standard basis!)
(e) Using the previous three parts, what is $[T]_{B \leftarrow B}$?

12. What is an eigenvector, geometrically speaking? What is the geometric meaning of an eigenvalue?

13. Pick your favorite $2 \times 2$ matrix. Compute its eigenvalues and eigenvalues. If you are not feeling imaginative, try $\begin{pmatrix} 1 & 2 \\ 4 & 7 \end{pmatrix}$.

14. What are the algebraic and geometric multiplicities of an eigenvalue? How do these relate to diagonalizability?

15. What is an inner product? In particular, what properties must a function $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$ satisfy in order to be an inner product? What is the norm induced by a given inner product?

16. Show that the function $\langle \cdot, \cdot \rangle : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ given by $\langle u, v \rangle = 7u_1v_1 + u_2v_2 + 9u_3v_3$ is an inner product.

17. Pick your favorite three points in $\mathbb{R}^2$ and write down the degree 1 and degree 2 least squares fits to these data.

18. How can you tell that two vectors are orthogonal?

19. What does it mean for a set of vectors to be orthonormal? How can you tell that a set of vectors are orthonormal?

20. Write down the Gram-Schmidt algorithm. In particular, suppose you have computed the first $k$ vectors in your orthonormal basis. How do you compute the next one?

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